AD NUMBER

AD802187

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution authorized to U.S. Gov't. agencies and their contractors; Critical Technology; SEP 1966. Other requests shall be referred to Army Missile Command, Attn: AMSMI-RS, Redstone Arsenal, AL 35809.

AUTHORITY

usamc ltr, 3 mar 1971

AD

802187

REPORT NO. RS-TR-66-4

SOLUTION OF TRANSIENT HEAT TRANSFER PROBLEMS FOR FLAT PLATES, CYLINDERS, AND SPHERES BY FINITE-DIFFERENCE METHODS WITH APPLICATION TO SURFACE RECESSION

by
R. Eppes, Jr.

September 1966

DISTRIBUTION LIMITED
SEE NOTICES PAGE

U.S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama

1 September 1966

Report No. RS-TR-66-4

SOLUTION OF TRANSIENT HEAT TRANSFER PROBLEMS FOR FLAT PLATES, CYLINDERS, AND SPHERES BY FINITE-DIFFERENCE METHODS WITH APPLICATION TO SURFACE RECESSION

by
R. Eppes, Jr.

DA Project No. 1L013001A91A
AMC Management Structure Code No. 501611844

DISTRIBUTION LIMITED
SEE NOTICES PAGE

Stress and Thermodynamics Analysis Branch
Structures and Mechanics Laboratory
Research and Development Directorate
U. S. Army Missile Command
Redstone Arsenal, Alabama 35809

ABSTRACT

Presented in this report are finite-difference heat-transfer equations for transient, radial heat flow in spheres and cylinders and for transient, one-dimensional heat flow in flat plates. The derived equations apply to structures before, during, and after surface recession for all three basic structure configurations and for several generic material skin combinations.

For each skin configuration, the accuracy of the finite-difference procedure, compared with exact analytical methods, depends on optimum selection of the calculation time increment and the incremental distance between temperature nodes in relation to the material thermal properties and on the closeness of the approximate temperature gradients to the true gradients. In addition to these common criteria, the magnitude of the surface recession rate in relation to the calculation time increment and temperature nodal point distance affects the accuracy of the finite-difference temperature results. When compared with exact solutions applicable to semi-infinite flat plates undergoing surface recession, the calculated finite-difference temperature gradients during recession are very accurate when the amount of material removed during a calculation time increment is equal to or less than one fourth of the selected distance increment between temperature nodes.

The cylindrical and spherical equations are presented for centripetal heat flow and surface recession. Two simple methods of converting the centripetal equations to the centrifugal form for applications to structures such as blast tubes, rocket motor combustion chambers, and nozzles are discussed. These two methods involve making a minor number of sign changes in the centripetal heat-flow equations.

Attractive features of the ablation-conduction method described in this report are the negligible increase in required computer time over a nonreceding case when all other parameters are identical. Secondly, the nonshifting temperature grid prevents confusion in interpreting computer results and readily lends itself to automatic plotting techniques.

ACKNOWLEDGMENT

The author wishes to acknowledge Messrs W. G. Burleson and R. A. Reynolds of the Structures and Mechanics Laboratory, Research and Development Directorate, Redstone Arsenal, Alabama, for the help, encouragement, and stimulating discussions of many problems that arose in the course of this investigation. Acknowledgement also goes to Mr. Donald A. Willoughby of Rohm and Haas Company, Redstone Division, Redstone Arsenal, Alabama, for reviewing and making many helpful suggestions to improve this report and to Mr. W. R. Sellers, Jr. of the U. S. Army Computation Center, Redstone Arsenal, Alabama, for his effort in programming the equations.

CONTENTS

SYMBOLS

A	-	Area.
θ	-	Angle in radians.
ϕ	-	Solid angle in steradians.
β		Dimensionless modulus $\left(\frac{k\Delta t}{\rho c \tau^2}\right)$.
R	-	Radius of cylindrical or spherical section.
$\sum \tau_n$	-	Distance from outer surface of cylindrical or spherical section to temperature point n (includes thickness of ablation into τ layer, a).
Δt	-	Time increment for computation.
T	-	Temperature.
k	-	Thermal conductivity of material.
c	-	Specific heat of material.
ρ	-	Density of material.
τ	-	Incremental thickness for each material.
q_{net}	-	Net heat flux at boundary.
a	-	Summation of ablation into any one τ $(0 \leq a \leq \tau)$, $\sum(\dot{a}\Delta t)$.
L	-	Length of cylinder (unity).
$\dot{a}$	-	Ablation or recession rate.
Z	-	$R - \sum \tau_2$.
B	-	Symbol notation defined after each use.
δ	-	Material thickness.

<u>Subscripts and Superscripts</u>

a	-	Material "A."
b	-	Material "B."
c	-	Material "C."
bs	-	Backside or internal surface.
o	-	External surface.

i	-	Internal surface.
'	-	Conditions existing after the lapse of one Δt.
n	-	Nodal point.
m	-	Melt condition.

Section I. INTRODUCTION

With the continuing use of reliable ablating reinforced plastics and subliming materials for efficient, economical, thermal protection of missile airframes and components, an accurate simple solution to problems of transient heat flow in solids experiencing a variable surface recession rate at one surface is required. This solution is not only necessary for flat plates but also for cylinders and spheres. General analytical solutions for structures undergoing surface recession are not available, and exact solutions are known only for special flat-plate cases.

The analysis of small hemispherically tipped vehicles can be more accurately calculated by a spherical program than a flat plate. Often many small semicylindrical leading edges, blast tubes, motor cases, and nozzles can better be assessed by a cylindrical procedure than by a flat-plate procedure.

A large majority of the materials used for thermal protection of supersonic missiles possess a very low thermal diffusivity. As a result one-dimensional heat flow in flat plates and radial heat flow in cylinders and spheres are sufficiently accurate even though the heat input usually varies along the exposed surface.

A numerical finite-difference method for heat flow before, during, and after surface recession on flat plates, cylinders, and spheres is described in this report. The equations derived for cylinders and spheres are for centripetal surface recession; however, two simple methods of using the same equations for centrifugal surface recession are discussed.

A brief comparison of calculated temperature distributions with exact results is discussed for special, ablating flat-plate cases. In addition to the criteria affecting the accuracy of finite-difference results for a plate with no recession, the accuracy of the numerical calculations for surface recession depends quite heavily on the judicious selection of the incremental node thickness and calculation time increment in terms of the actual surface recession rate.

The advantages of the ablation-conduction method presented in this report are the simplicity of its formulation, the versatility of the

boundary conditions (variable recession rate, numerous material combinations, and variable thermal properties),* and the short computer time required. For the same structural arrangement and identical selections of variables such as node thickness and calculation time increment, a recession computation requires a negligible increase in computer time over the nonrecession case.

*Temperature dependent approximations for specific heat and thermal conductivity.

Section II. HEAT CONDUCTION WITHOUT SURFACE RECESSION

1. Transient, One-Dimensional Heat Transfer for Flat Plates

One-dimensional, flat-plate heat transfer in a homogeneous material may be determined by solving heat balance equations at the exposed surface, unexposed surface, interior nodes, and interfaces. The forward finite-difference method was used to solve the heat balance equations. It was assumed that the incremental thickness (τ) can be selected small enough to give accurate temperature gradients between adjacent nodes and that the incremental time (Δt) is small enough to neglect any effect on regions more than one τ from the node in question. The stability criteria for the forward finite difference equations can be found in Report No. RS-TR-65-1.[1]

a. Thick Material

(1) Exterior Surface. From Figure 1 the heat balance at tl exposed surface is

$$q_{net_o} - q_{\substack{cond \\ 1 \rightarrow 2}} = q_{\substack{stored \\ 1}} \tag{1}$$

where

q_{net_o} = net heat received per unit area

$$q_{\substack{cond \\ 1 \rightarrow 2}} = k_a A \frac{(T_1 - T_2)}{\tau_a}$$

$$q_{\substack{stored \\ 1}} = \rho_a c_a \frac{\tau_a}{2} A \frac{(T_1' - T_1)}{\Delta t} .$$

The area, A, is uniform for one-dimensional, flat-plate heat transfer. Rewriting Equation (1) we have

$$q_{net_o} - \frac{k_a}{\tau_a} (T_1 - T_2) = \rho_a c_a \frac{\tau_a}{2} \frac{(T_1' - T_1)}{\Delta t} \tag{2}$$

[1]U. S. Army Missile Command, Redstone Arsenal, Alabama, SOLUTION OF TRANSIENT HEAT TRANSFER PROBLEMS FOR FLAT PLATES, CYLINDERS, AND SPHERES BY FINITE-DIFFERENCE METHODS by W. G. Burleson and R. Eppes, Jr., 15 March 1965, Report No. RS-TR-65-1 (Unclassified Report) AD 461 662.

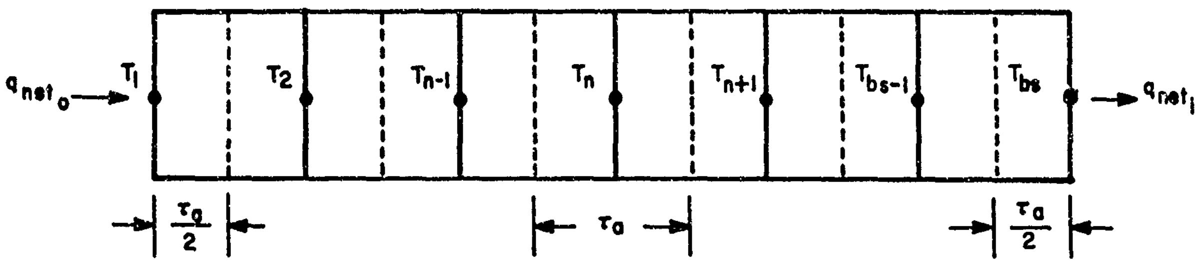


Figure 1.

or

$$T_1' = T_1 + 2\beta_a \left(T_2 - T_1\right) + \frac{2\, q_{net_o}\, \Delta t}{\rho_a\, c_a\, \tau_a} \tag{3}$$

where

$$\beta_a = \frac{k_a\, \Delta t}{\rho_a\, c_a\, \tau_a^2} \quad .$$

(2) Interior Node. The energy balance at any interior point n of a homogeneous wall (Figure 1) may be written

$$\underset{n-1\rightarrow n}{q_{cond}} + \underset{n+1\rightarrow n}{q_{cond}} = \underset{n}{q_{stored}} \tag{4}$$

or

$$\frac{k_a}{\tau_a}\left(T_{n-1} - T_n\right) + \frac{k_a}{\tau_a}\left(T_{n+1} - T_n\right) = \rho_a\, c_a\, \tau_a \frac{\left(T_n' - T_n\right)}{\Delta t} \quad . \tag{5}$$

Solving for T_n' with

$$\beta_a = \frac{k_a\, \Delta t}{\rho_a\, c_a\, \tau_a^2}$$

$$T_n' = T_n\left(1 - 2\beta_a\right) + \beta_a\left(T_{n-1} + T_{n+1}\right) \quad . \tag{6}$$

(3) Backside Surface. The energy balance at the backside surface (Figure 1), T_{bs}, may be written as

$$\underset{bs-1\rightarrow bs}{q_{cond}} - q_{net_i} = \underset{bs}{q_{stored}} \tag{7}$$

or

$$\frac{k_a}{\tau_a}\left(T_{bs-1} - T_{bs}\right) - q_{net_i} = \rho_a\, c_a \frac{\tau_a}{2} \frac{\left(T'_{bs} - T_{bs}\right)}{\Delta t} \quad . \tag{8}$$

Rearrange and solve for T'_{bs} with

$$\beta_a = \frac{k_a\, \Delta t}{\rho_a\, c_a\, \tau_a^{\,2}}$$

$$T'_{bs} = T_{bs} + 2\beta_a\left(T_{bs-1} - T_{bs}\right) - q_{net_i}\frac{2\,\Delta t}{\rho_a\, c_a\, \tau_a} \quad . \tag{9}$$

b. Thick-Thick Material

At the interface between material "A" and "B" (T_n, Figure 2), the energy balance is

$$\underset{n-1\rightarrow n}{q_{cond}} + \underset{n+1\rightarrow n}{q_{cond}} = \underset{n}{q_{stored}}$$

or

$$\frac{k_a}{\tau_a}\left(T_{n-1} - T_n\right) + \frac{k_b}{\tau_b}\left(T_{n+1} - T_n\right)$$

$$= \left(\rho_a\, c_a \frac{\tau_a}{2} + \rho_b\, c_b \frac{\tau_b}{2}\right) \frac{\left(T'_n - T_n\right)}{\Delta t} \, . \tag{11}$$

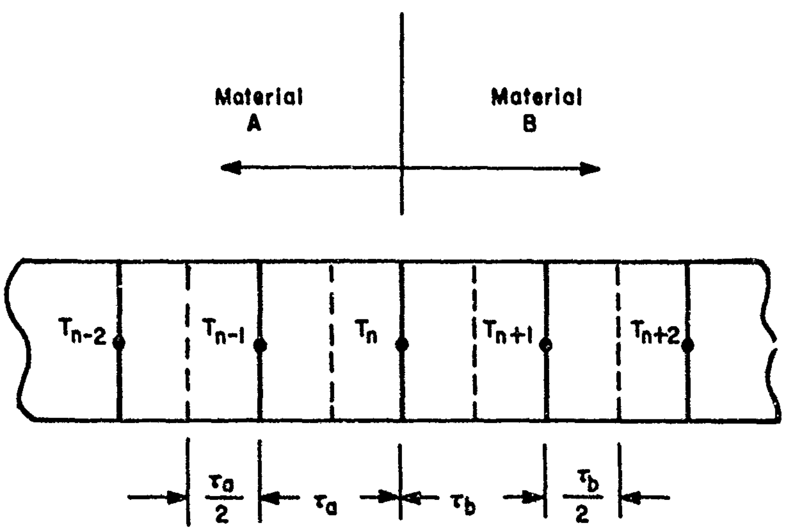


Figure 2.

Rearrange and solve for T_n'

$$T_n' = T_n + \left(T_{n-1} - T_n\right) \frac{2\, k_a\, \Delta t}{\tau_a\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}$$

$$+ \left(T_{n+1} - T_n\right) \frac{2\, k_b\, \Delta t}{\tau_b\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)} . \qquad (12)$$

c. Thin-Thick Material

At the exposed surface (Figure 3) the energy balance for the thermally thin-thermally thick interface is

$$q_{net_o} + q_{\underset{2 \to 1}{cond}} = q_{\underset{1}{stored}} \qquad (13)$$

or

$$q_{net_o} + \frac{k_b}{\tau_b}\left(T_2 - T_1\right) = \left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b \frac{\tau_b}{2}\right) \frac{\left(T_1' - T_1\right)}{\Delta t} . \qquad (14)$$

Rearrange and solve for T_1'

$$T_1' = T_1 + \frac{q_{net_o}\, \Delta t}{\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b \frac{\tau_b}{2}\right)} + \frac{k_b}{\tau_b} \frac{\Delta t\left(T_2 - T_1\right)}{\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b \frac{\tau_b}{2}\right)} . \qquad (15)$$

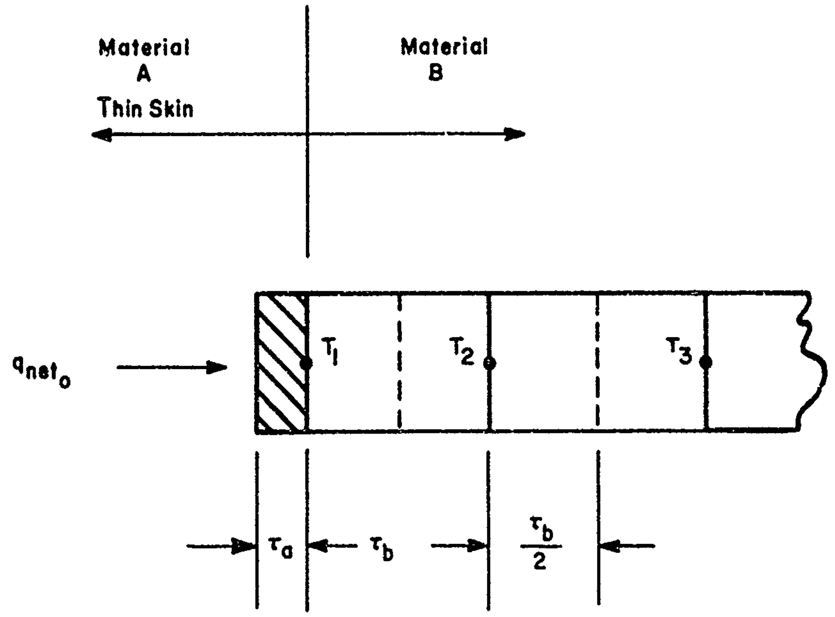


Figure 3.

d. Thick-Thin Material

The energy balance at the backside surface (Figure 4), T_{bs}, may be written as

$$\underset{bs-1 \rightarrow bs}{q_{cond}} - q_{net_i} = \underset{bs}{q_{stored}} \tag{16}$$

or

$$\frac{k_a}{\tau_a} \left(T_{bs-1} - T_{bs}\right) - q_{net_i} = \left(\rho_a \, c_a \frac{\tau_a}{2} + \rho_b \, c_b \, \tau_b\right) \frac{\left(T'_{bs} - T_{bs}\right)}{\Delta t} . \tag{17}$$

Rearrange and solve for T'_{bs}.

$$T'_{bs} = T_{bs} + \frac{k_a \, \Delta t \left(T_{bs-1} - T_{bs}\right)}{\tau_a \left(\rho_a \, c_a \frac{\tau_a}{2} + \rho_b \, c_b \, \tau_b\right)} - \frac{q_{net_i} \, \Delta t}{\left(\rho_a \, c_a \frac{\tau_a}{2} + \rho_b \, c_b \, \tau_b\right)} . \tag{18}$$

e. Thick-Thin-Thick Material

At the interface between material "A" and "C" (T_n, Figure 5), the energy balance for the thermally thin material "B" is

$$\underset{n-1 \rightarrow n}{q_{cond}} + \underset{n+1 \rightarrow n}{q_{cond}} = \underset{n}{q_{stored}} \tag{19}$$

or

$$\frac{k_a}{\tau_a} \left(T_{n-1} - T_n\right) + \frac{k_c}{\tau_c} \left(T_{n+1} - T_n\right) = \left(\rho_a \, c_a \frac{\tau_a}{2} + \rho_b \, c_b \, \tau_b + \rho_c \, c_c \frac{\tau_c}{2}\right) \frac{\left(T'_n - T_n\right)}{\Delta t} . \tag{20}$$

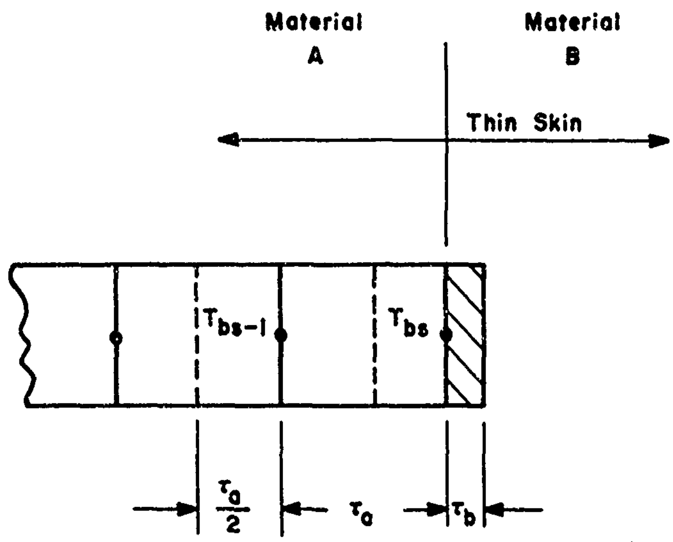


Figure 4

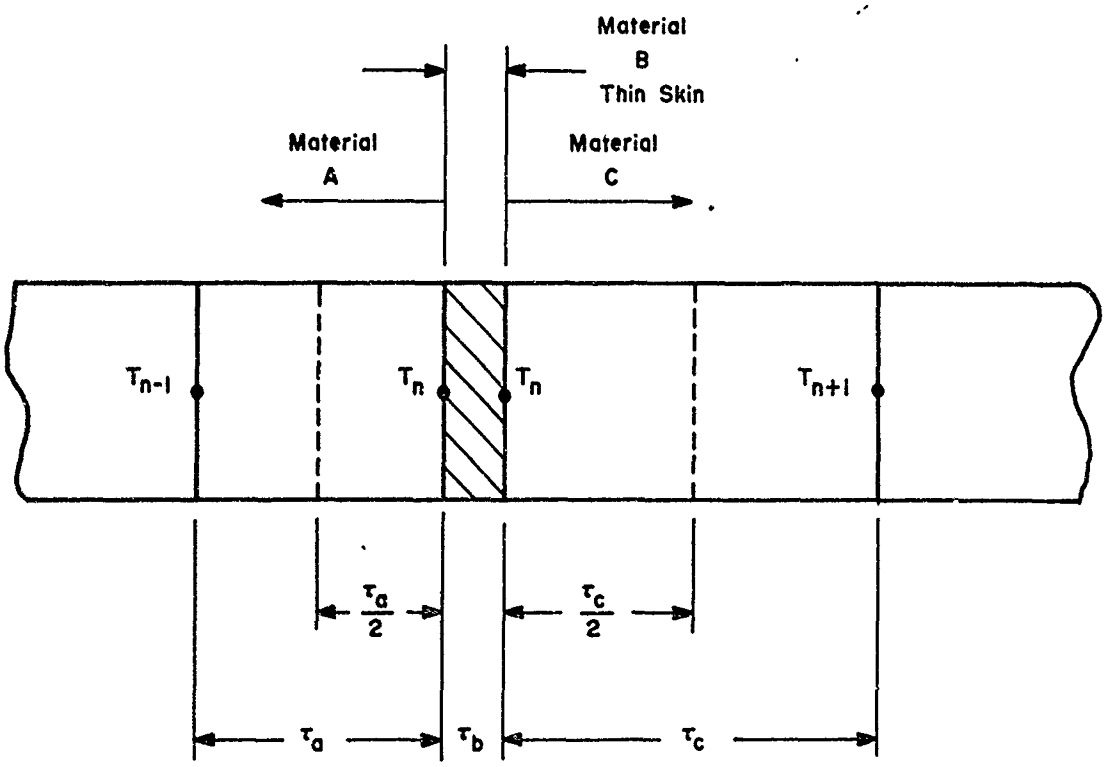


Figure 5

Rearrange and solve for T_n'

$$T_n' = T_n + \frac{k_a \Delta t \left(T_{n-1} - T_n\right)}{\tau_a \left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right)}$$

$$+ \frac{k_c \Delta t \left(T_{n+1} - T_n\right)}{\tau_c \left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right)} . \qquad (21)$$

2. **Transient, Radial Heat Transfer for Cylinders**

a. Thick Material

(1) Exterior Surface. Consider a cylindrical segment heated as shown in Figure 6. From the energy balance at the peripheral surface

$$q_{in} A_1 - q_{\underset{1 \to 2}{out}} A_2 = q_{\underset{1}{stored}} A_3 \qquad (22)$$

where

$$q_{in} = q_{net_o}, \quad A_1 = R\theta L$$

$$q_{\underset{1 \to 2}{out}} = \frac{k_a}{\tau_a} \left(T_1 - T_2\right), \quad A_2 = \left(R - \frac{\tau_a}{2}\right) \theta L$$

$$q_{\underset{1}{stored}} = \rho_a c_a \frac{\tau_a}{2} \frac{\left(T_1' - T_1\right)}{\Delta t}, \quad A_3 = \left(R - \frac{\tau_a}{4}\right) \theta L$$

or

$$R\theta L \, q_{net_o} - \left(R - \frac{\tau_a}{2}\right) \theta L \frac{k_a}{\tau_a} \left(T_1 - T_2\right)$$

$$= \left(R - \frac{\tau_a}{4}\right) \theta L \left(\rho_a c_a \frac{\tau_a}{2}\right) \frac{\left(T_1' - T_1\right)}{\Delta t} . \qquad (23)$$

Let

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad ;$$

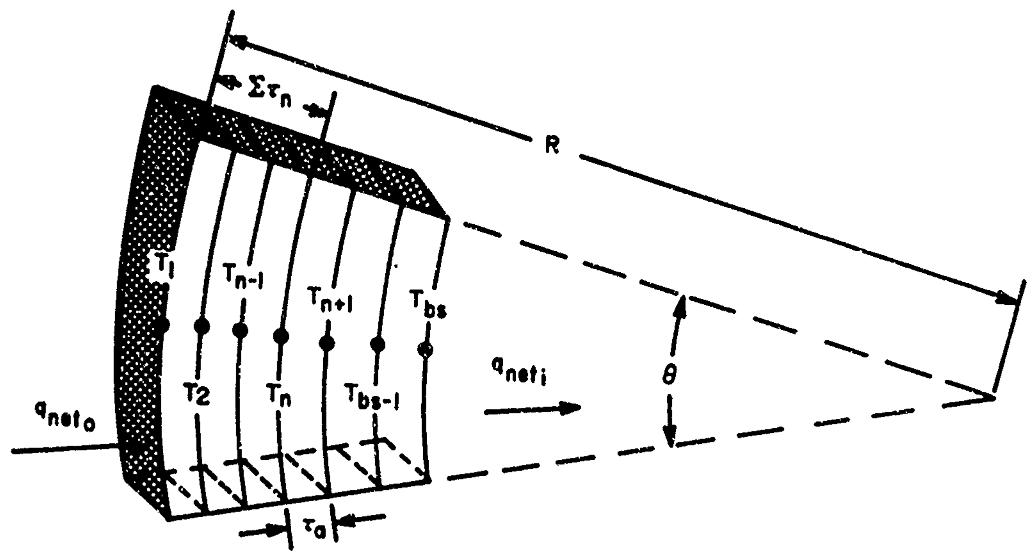


Figure 6.

rearrange; and solve for T_1'

$$T_1' = T_1 + \frac{q_{neto}\, R\, \Delta t}{\left(R - \frac{\tau_a}{4}\right)\left(\rho_a\, c_a\, \frac{\tau_a}{2}\right)} - 2\beta_a \left(\frac{R - \frac{\tau_a}{2}}{R - \frac{\tau_a}{4}}\right)\left(T_1 - T_2\right). \tag{24}$$

(2) Interior Node. The energy balance at any interior point n of a homogeneous wall (Figure 6) may be written

$$\underset{n-1 \to n}{q_{cond}}\; A_1 + \underset{n+1 \to n}{q_{cond}}\; A_2 = \underset{n}{q_{stored}}\; A_3 \tag{25}$$

where

$$A_1 = \left(R - \sum \tau_n + \frac{\tau_a}{2}\right) \theta L$$

$$A_2 = \left(R - \sum \tau_n - \frac{\tau_a}{2}\right) \theta L$$

$$A_3 = \left(R - \sum \tau_n\right) \theta L$$

or

$$\theta L \frac{k_a}{\tau_a}\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)\left(T_{n-1} - T_n\right) + \theta L \frac{k_a}{\tau_a}\left(R - \sum \tau_n - \frac{\tau_a}{2}\right)\left(T_{n+1} - T_n\right)$$

$$= \theta L \left(R - \sum \tau_n\right) \rho_a\, c_a\, \tau_a\, \frac{\left(T_n' - T_n\right)}{\Delta t}. \tag{26}$$

Let

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad ;$$

rearrange; and solve for T_n'

$$T_n' = T_n + \beta_a \left(\frac{R - \sum \tau_n + \frac{\tau_a}{2}}{R - \Sigma \tau_n} \right) \left(T_{n-1} - T_n \right)$$
$$+ \beta_a \left(\frac{R - \sum \tau_n - \frac{\tau_a}{2}}{R - \Sigma \tau_n} \right) \left(T_{n+1} - T_n \right). \tag{27}$$

(3) Backside Surface. The energy balance at the backside surface (Figure 6), T_{bs}, may be written as

$$\underset{bs-1 \rightarrow bs}{q_{cond}} A_1 - q_{net_i} A_2 = \underset{bs}{q_{stored}} A_3 \tag{28}$$

where

$$\underset{bs-1 \rightarrow bs}{q_{cond}} A_1 = \frac{k_a}{\tau_a} \left(T_{bs-1} - T_{bs} \right) \left(R - \sum \tau_{bs} + \frac{\tau_a}{2} \right) \theta L$$

$$q_{net_i} A_2 = q_{net_i} \left(R - \sum \tau_{bs} \right) \theta L$$

$$\underset{bs}{q_{stored}} A_3 = \rho_a c_a \frac{\tau_a}{2} \frac{\left(T_{bs}' - T_{bs} \right)}{\Delta t} \left(R - \sum \tau_{bs} + \frac{\tau_a}{4} \right) \theta L$$

or

$$\theta L \frac{k_a}{\tau_a} \left(R - \sum \tau_{bs} + \frac{\tau_a}{2} \right) \left(T_{bs-1} - T_{bs} \right) - \theta L \, q_{net_i} \left(R - \sum \tau_{bs} \right)$$
$$= \theta L \rho_a c_a \frac{\tau_a}{2} \left(R - \sum \tau_{bs} + \frac{\tau_a}{4} \right) \frac{\left(T_{bs}' - T_{bs} \right)}{\Delta t}. \tag{29}$$

Let

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad ;$$

rearrange; and solve for T'_{bs}

$$T'_{bs} = T_{bs} + 2\beta_a \left(\frac{R - \sum \tau_{bs} + \frac{\tau_a}{2}}{R - \sum \tau_{bs} + \frac{\tau_a}{4}} \right) \left(T_{bs-1} - T_{bs} \right)$$

$$- q_{net_i} \frac{2\Delta t}{\rho_a c_a \tau_a} \left(\frac{R - \sum \tau_{bs}}{R - \sum \tau_{bs} + \frac{\tau_a}{4}} \right) . \tag{30}$$

b. Thick-Thick Material

At the interface between material "A" and "B" (T_n, Figure 7), the energy balance is

$$q_{\substack{cond \\ n-1 \to n}} A_1 + q_{\substack{cond \\ n+1 \to n}} A_2 = q_{\substack{stored \\ n}} A_3 \tag{31}$$

where

$$q_{\substack{cond \\ n-1 \to n}} A_1 = \frac{k_a}{\tau_a} \left(T_{n-1} - T_n \right) \left(R - \sum \tau_n + \frac{\tau_a}{2} \right) \theta L$$

$$q_{\substack{cond \\ n+1 \to n}} A_2 = \frac{k_b}{\tau_b} \left(T_{n+1} - T_n \right) \left(R - \sum \tau_n - \frac{\tau_b}{2} \right) \theta L$$

$$q_{\substack{stored \\ n}} A_3 = \left[\left(R - \sum \tau_n + \frac{\tau_a}{4} \right) \rho_a c_a \frac{\tau_a}{2} + \left(R - \sum \tau_n - \frac{\tau_b}{4} \right) \rho_b c_b \frac{\tau_b}{2} \right] \frac{\left(T'_n - T_n \right)}{\Delta t} \quad \theta L$$

or

$$\theta L \frac{k_a}{\tau_a} \left(R - \sum \tau_n + \frac{\tau_a}{2} \right) \left(T_{n-1} - T_n \right) + \theta L \frac{k_b}{\tau_b} \left(R - \sum \tau_n - \frac{\tau_b}{2} \right) \left(T_{n+1} - T_n \right)$$

$$= \theta L \left[\left(R - \sum \tau_n + \frac{\tau_a}{4} \right) \rho_a c_a \frac{\tau_a}{2} + \left(R - \sum \tau_n - \frac{\tau_b}{4} \right) \rho_b c_b \frac{\tau_b}{2} \right] \frac{\left(T'_n - T_n \right)}{\Delta t} . \tag{32}$$

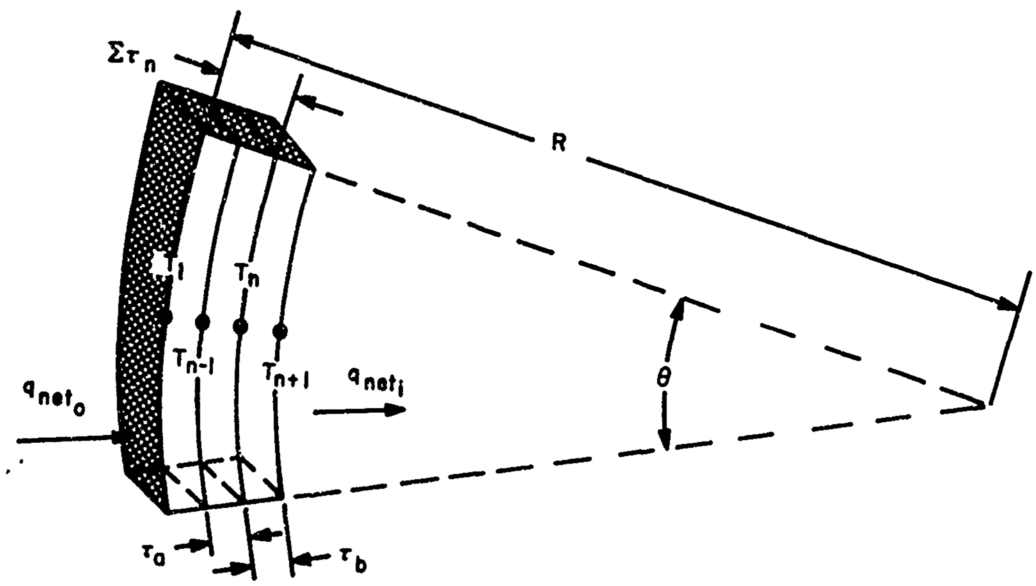


Figure 7.

Solving for T_n'

$$T_n' = T_n + \frac{2\,k_a\,\Delta t\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)\left(T_{n-1} - T_n\right)}{\tau_a\left[\left(R - \sum \tau_n + \frac{\tau_a}{4}\right)\rho_a\,c_a\,\tau_a + \left(R - \sum \tau_n - \frac{\tau_b}{4}\right)\rho_b\,c_b\,\tau_b\right]}$$

$$+ \frac{2\,k_b\,\Delta t\left(R - \sum \tau_n - \frac{\tau_b}{4}\right)\left(T_{n+1} - T_n\right)}{\tau_b\left[\left(R - \sum \tau_n + \frac{\tau_a}{4}\right)\rho_a\,c_a\,\tau_a + \left(R - \sum \tau_n - \frac{\tau_b}{4}\right)\rho_b\,c_b\,\tau_b\right]} \cdot \qquad (33)$$

c. Thin-Thick Material

At the exposed surface (Figure 8) the energy balance for the thermally thin-thermally thick interface is

$$q_{net_o}\,A_1 + q_{\underset{2\rightarrow 1}{cond}}\,A_2 = q_{\underset{1}{stored}}\,A_3 \qquad (34)$$

where

$$q_{net_o}\,A_1 = q_{net_o}\,R\,\theta L$$

$$q_{\underset{2\rightarrow 1}{cond}}\,A_2 \quad \frac{k_b}{\tau_b}\left(T_2 - T_1\right)\left(R - \tau_a - \frac{\tau_b}{2}\right)\theta L$$

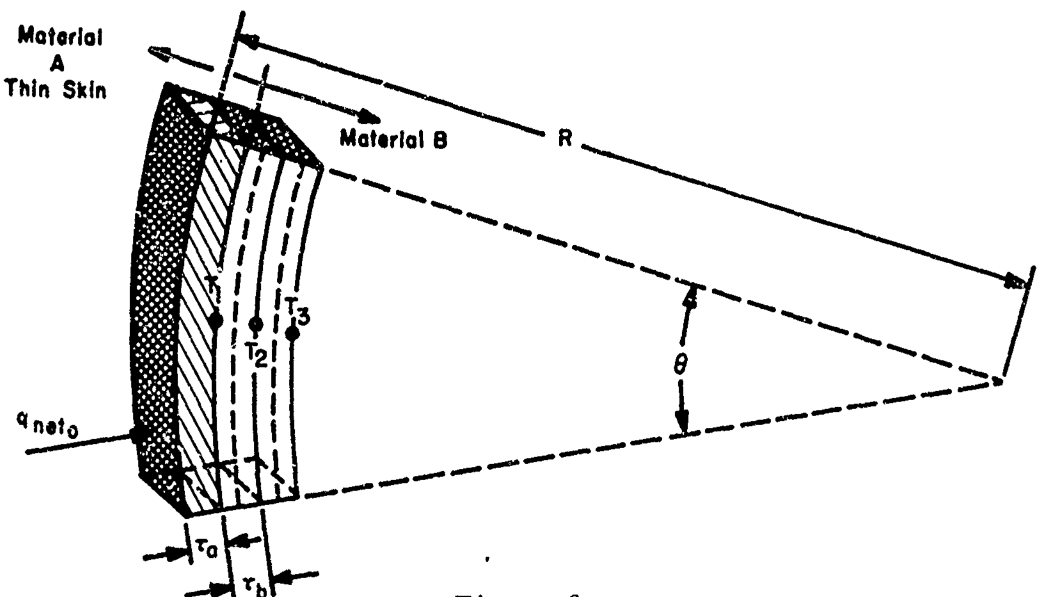


Figure 8.

$$q_{stored_1} A_3 = \left[\rho_a c_a \tau_a \left(R - \frac{\tau_a}{2}\right) + \rho_b c_b \frac{\tau_b}{2}\left(R - \tau_a - \frac{\tau_b}{4}\right)\right] \frac{\left(T_1' - T_1\right)}{\Delta t} \quad \theta L$$

or

$$\theta L\, R\, q_{net_o} + \theta L \frac{k_b}{\tau_b}\left(R - \tau_a - \frac{\tau_b}{2}\right)\left(T_2 - T_1\right) = \theta L\left[\rho_a c_a \tau_a \left(R - \frac{\tau_a}{2}\right) + \rho_b c_b \frac{\tau_b}{2}\left(R - \tau_a - \frac{\tau_b}{4}\right)\right] \frac{\left(T_1' - T_1\right)}{\Delta t} . \qquad (35)$$

Rearrange and solve for T_1'

$$T_1' = T_1 + \frac{q_{net_o}\, \Delta t\, R}{\left[\rho_a c_a \tau_a \left(R - \frac{\tau_a}{2}\right) + \rho_b c_b \frac{\tau_b}{2}\left(R - \tau_a - \frac{\tau_b}{4}\right)\right]} + \frac{k_b\, \Delta t \left(R - \tau_a - \frac{\tau_b}{2}\right)\left(T_2 - T_1\right)}{\tau_b\left[\rho_a c_a \tau_a \left(R - \frac{\tau_a}{2}\right) + \rho_b c_b \frac{\tau_b}{2}\left(R - \tau_a - \frac{\tau_b}{4}\right)\right]} . \qquad (36)$$

d. Thick-Thin Material

The energy balance at the backside surface (Figure 9), T_{bs}, may be written as

$$q_{\underset{bs-1\rightarrow bs}{cond}} A_1 - q_{net_i} A_2 = q_{\underset{bs}{stored}} A_3 \tag{37}$$

where

$$q_{\underset{bs-1\rightarrow bs}{cond}} A_1 = \frac{k_a}{\tau_a}\left(T_{bs-1} - T_{bs}\right)\left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)\theta L$$

$$q_{net_i} A_2 = q_{net_i}\left(R - \sum \tau_{bs} - \tau_b\right)\theta L$$

$$q_{\underset{bs}{stored}} A_3 = \left[\rho_a\, c_a \frac{\tau_a}{2}\left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)\right.$$

$$\left. + \rho_b\, c_b\, \tau_b\left(R - \sum \tau_{bs} - \frac{\tau_b}{2}\right)\right]\frac{\left(T'_{bs} - T_{bs}\right)}{\Delta t}\,\theta L$$

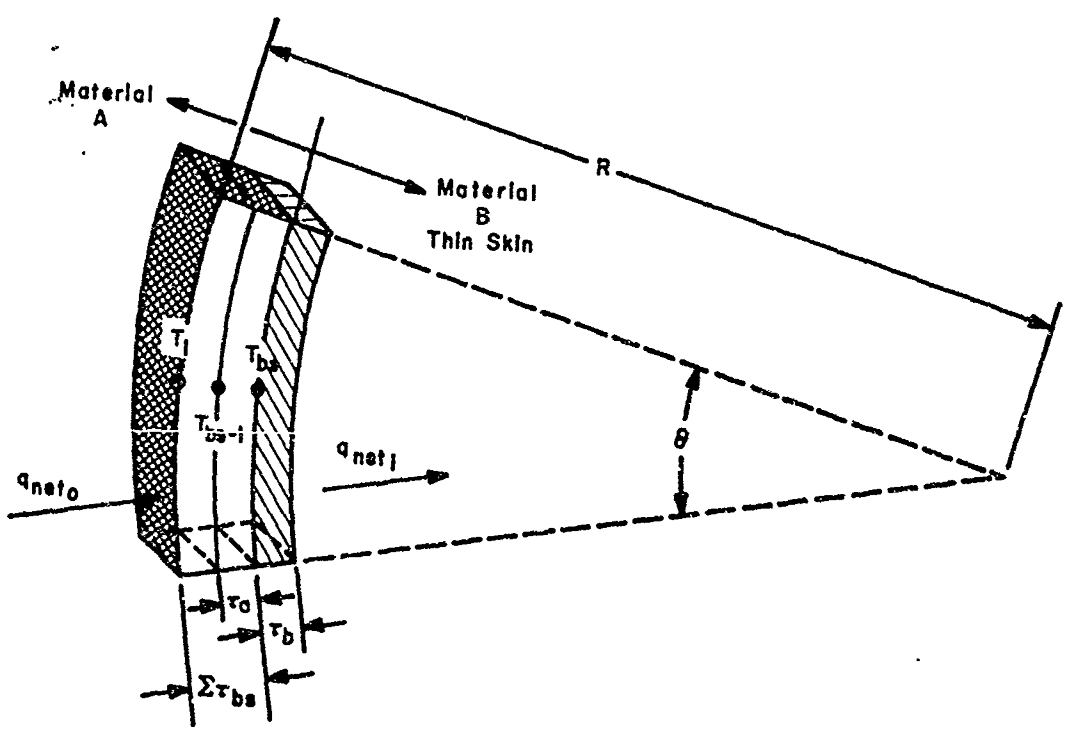


Figure 9.

or

$$\theta L \frac{k_a}{\tau_a}\left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)\left(T_{bs-1} - T_{bs}\right) - \theta L\, q_{net_i}\left(R - \sum \tau_{bs} - \tau_b\right)$$

$$- \theta L \left[\rho_a c_a \frac{\tau_a}{2}\left(R \sum \tau_{bs} + \frac{\tau_a}{4}\right)\right.$$

$$\left. + \rho_b c_b \tau_b \left(R - \sum \tau_{bs} - \frac{\tau_b}{2}\right)\right] \frac{\left(T'_{bs} - T_{bs}\right)}{\Delta t}. \tag{38}$$

Rearrange and solve for T'_{bs}

$$T'_{bs} - T_{bs} + \frac{k_a \Delta t \left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)\left(T_{bs-1} - T_{bs}\right)}{\tau_a\left[\rho_a c_a \frac{\tau_a}{2}\left(R \sum \tau_{bs} + \frac{\tau_a}{4}\right) + \rho_b c_b \tau_b \left(R \sum \tau_{bs} - \frac{\tau_b}{2}\right)\right]}$$

$$- \frac{q_{net_i} \Delta t \left(R - \sum \tau_{bs} - \tau_b\right)}{\left[\rho_a c_a \frac{\tau_a}{2}\left(R \sum \tau_{bs} + \frac{\tau_a}{4}\right) + \rho_b c_b \tau_b \left(R \sum \tau_{bs} - \frac{\tau_b}{2}\right)\right]}. \tag{39}$$

e. Thick-Thin-Thick Material

At the interface between material "A" and "C" (T_n, Figure 10), the energy balance for the thermally thin material "B" is

$$q_{\underset{n-1 \to n}{cond}} A_1 + q_{\underset{n+1 \to n}{cond}} A_2 = q_{\underset{n}{stored}} A_3 \tag{40}$$

where

$$q_{\underset{n-1 \to n}{cond}} A_1 = \frac{k_a}{\tau_a}\left(T_{n-1} - T_n\right)\left(R - \sum \tau_n + \frac{\tau_a}{2}\right) \theta L$$

$$q_{\underset{n+1 \to n}{cond}} A_2 = \frac{k_c}{\tau_c}\left(T_{n+1} - T_n\right)\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{2}\right) \theta L$$

$$q_{\underset{n}{stored}} A_3 = \left[\rho_a c_a \frac{\tau_a}{2}\left(R - \sum \tau_n + \frac{\tau_a}{4}\right) + \rho_b c_b \tau_b \left(R - \sum \tau_n - \frac{\tau_b}{2}\right)\right.$$

$$\left. + \rho_c c_c \frac{\tau_c}{2}\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{4}\right)\right] \frac{\left(T'_n - T_n\right)}{\Delta t} \theta L$$

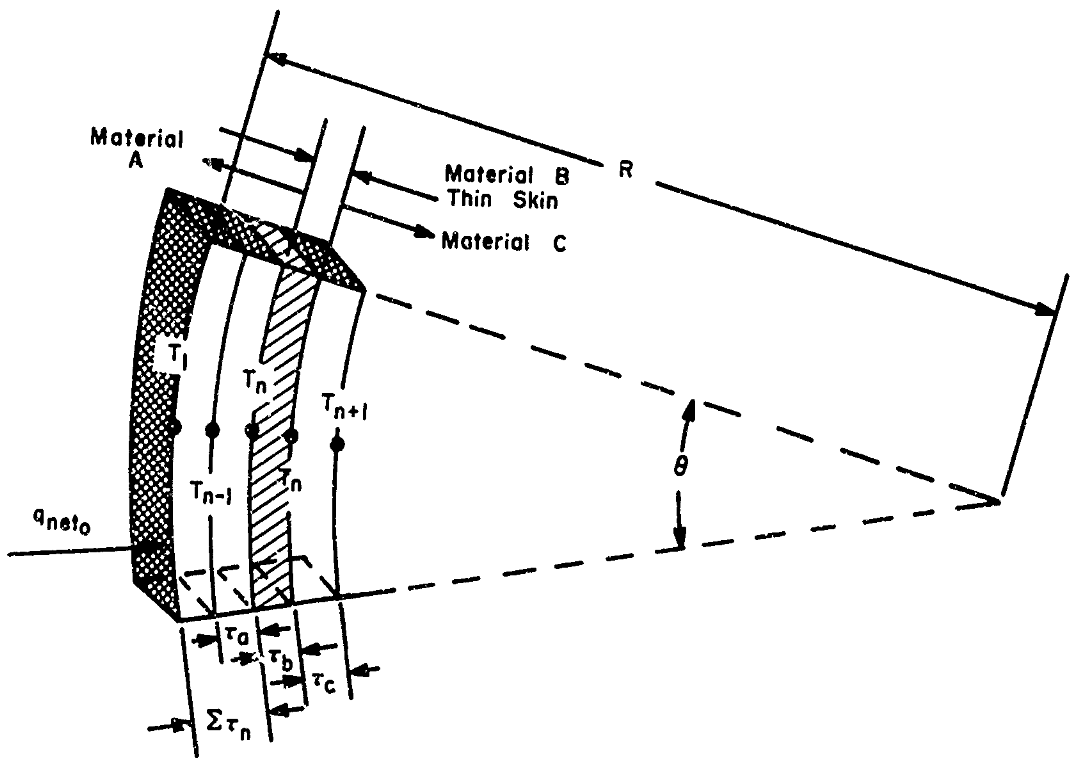


Figure 10.

or

$$\theta L \frac{k_a}{\tau_a}\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)\left(T_{n-1} - T_n\right) + \theta L \frac{k_c}{\tau_c}\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{2}\right)\left(T_{n+1} - T_n\right)$$

$$= \theta L \left[\rho_a c_a \frac{\tau_a}{2}\left(R - \sum \tau_n + \frac{\tau_a}{4}\right) + \rho_b c_b \tau_b \left(R - \sum \tau_b - \frac{\tau_b}{2}\right) + \rho_c c_c \frac{\tau_c}{2}\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{4}\right)\right] \frac{\left(T_n' - T_n\right)}{\Delta t} \quad . \qquad (41)$$

Rearrange and solve for T_n'

$$T_n' = T_n + \frac{k_a \Delta t \left(R - \sum \tau_n + \frac{\tau_a}{2}\right)\left(T_{n-1} - T_n\right)}{\tau_a \left[\rho_a c_a \frac{\tau_a}{2}\left(R - \sum \tau_n + \frac{\tau_a}{4}\right) + \rho_b c_b \tau_b \left(R - \sum \tau_n - \frac{\tau_b}{2}\right) + \rho_c c_c \frac{\tau_c}{2}\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{4}\right)\right]}$$

$$+ \frac{k_c \Delta t \left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{2}\right)\left(T_{n+1} - T_n\right)}{\tau_c \left[\rho_a c_a \frac{\tau_a}{2}\left(R - \sum \tau_n + \frac{\tau_a}{4}\right) + \rho_b c_b \tau_b \left(R - \sum \tau_n - \frac{\tau_b}{2}\right) + \rho_c c_c \frac{\tau_c}{2}\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{4}\right)\right]} \; . \qquad (42)$$

3. Transient, Radial Heat Transfer for Spheres

a. Thick Material

(1) External Surface. Consider a spherical segment heated as shown in Figure 11. From the energy balance at the peripheral surface

$$q_{net_o} A_1 - q_{cond_{1\to 2}} A_2 = q_{stored_1} A_3 \tag{43}$$

where

$$q_{net_o} A_1 = q_{net_o} R^2 \phi$$

$$q_{cond_{1\to 2}} A_2 = \frac{k_a}{\tau_a}\left(T_1 - T_2\right)\left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right]\phi$$

$$q_{stored_1} A_3 = \rho_a c_a \frac{\tau_a}{2} \frac{\left(T_1' - T_1\right)}{\Delta t}\left[\left(R - \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]\phi$$

or

$$\phi R^2 q_{net_o} - \phi\left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right]\frac{k_a}{\tau_a}\left(T_1 - T_2\right)$$

$$= \phi\left[\left(R - \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]\rho_a c_a \frac{\tau_a}{2}\frac{\left(T_1' - T_1\right)}{\Delta t} \ . \tag{44}$$

(The average area terms are derived in detail in Report No. RS-TR-65-1.)[2]

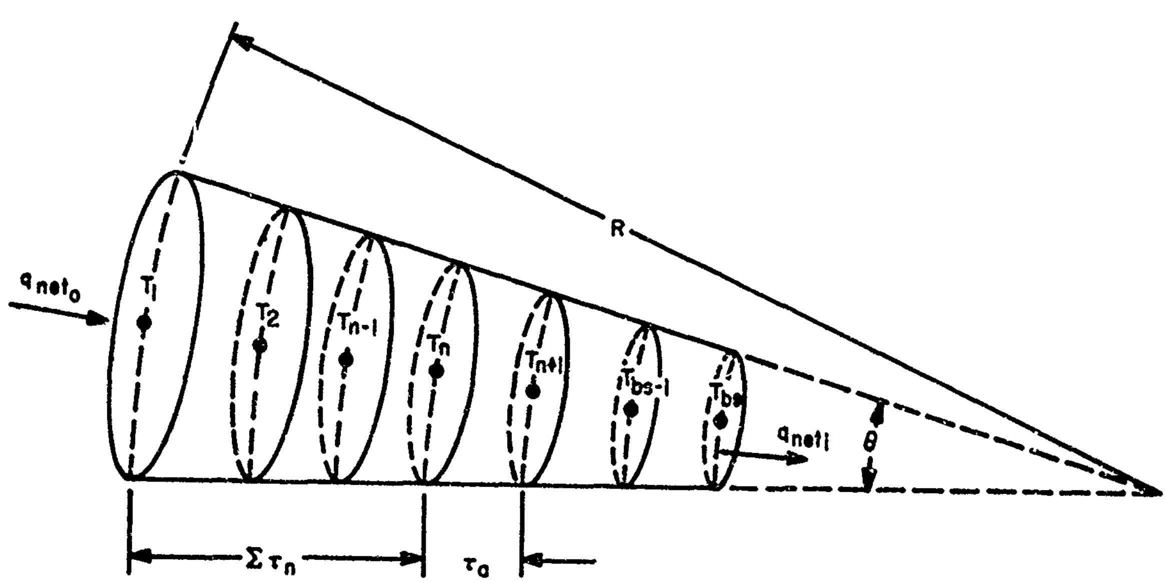


Figure 11.

[2] Burleson and Eppes, loc. cit.

Let

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^{\,2}} \quad ;$$

rearrange; and solve for T_1'

$$T_1' = T_1 + \frac{q_{net_o} \, R^2 \, \Delta t}{\left[\left(R - \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \rho_a \, c_a \, \frac{\tau_a}{2}} - 2\beta_a \frac{\left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \left(T_1 - T_2\right)}{\left[\left(R - \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]} \quad . \tag{45}$$

(2) Interior Node. The energy balance at any interior point n of a homogeneous wall (Figure 11) may be written

$$\underset{n-1 \to n}{q_{cond}} A_1 + \underset{n+1 \to n}{q_{cond}} A_2 = \underset{n}{q_{stored}} A_3 \tag{46}$$

where

$$\underset{n-1 \to n}{q_{cond}} A_1 = \frac{k_a}{\tau_a} \left(T_{n-1} - T_n\right) \left[\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \phi$$

$$\underset{n+1 \to n}{q_{cond}} A_2 = \frac{k_a}{\tau_a} \left(T_{n+1} - T_n\right) \left[\left(R - \sum \tau_n - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \phi$$

$$\underset{n}{q_{stored}} A_3 = \rho_a \, c_a \, \tau_a \, \frac{\left(T_n' - T_n\right)}{\Delta t} \left[\left(R - \sum \tau_n\right)^2 + \frac{\tau_a^2}{12}\right] \phi$$

or

$$\phi \left[\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \frac{k_a}{\tau_a} \left(T_{n-1} - T_n\right) + \phi \left[\left(R - \sum \tau_n - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \frac{k_a}{\tau_a} \left(T_{n+1} - T_n\right) = \phi \left[\left(R - \sum \tau_n\right)^2 + \frac{\tau_a^2}{12}\right] \rho_a \, c_a \, \tau_a \, \frac{\left(T_n' - T_n\right)}{\Delta t} \quad . \tag{47}$$

Let

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad ;$$

rearrange; and solve for T_n'

$$T_n' = T_n + \beta_a \left[\frac{\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{\left(R - \sum \tau_n\right)^2 + \frac{\tau_a^2}{12}} \right] \left(T_{n-1} - T_n\right)$$

$$+ \beta_a \left[\frac{\left(R - \sum \tau_n - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{\left(R - \sum \tau_n\right)^2 + \frac{\tau_a^2}{12}} \right] \left(T_{n+1} - T_n\right) . \qquad (48)$$

(3) <u>Backside Surface.</u> The energy balance at the backside surface (Figure 11), T_{bs}, may be written as

$$\underset{bs-1 \rightarrow bs}{q_{cond}} A_1 - q_{net_i} A_2 = \underset{bs}{q_{stored}} A_3 \qquad (49)$$

where

$$\underset{bs-1 \rightarrow bs}{q_{cond}} A_1 = \frac{k_a}{\tau_a} \left(T_{bs-1} - T_{bs}\right) \left[\left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \phi$$

$$q_{net_i} A_2 = q_{net_i} \left[\left(R - \sum \tau_{bs}\right)^2\right] \phi$$

$$\underset{bs}{q_{stored}} A_3 = \rho_a \, c_a \, \frac{\tau_a}{2} \, \frac{\left(T_{bs}' - T_{bs}\right)}{\Delta t} \left[\left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \phi$$

or

$$\phi \left[\left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \frac{k_a}{\tau_a} \left(T_{bs-1} - T_{bs}\right) - q_{net_i} \, \phi \left[\left(R - \sum \tau_{bs}\right)^2\right]$$

$$= \phi \left[\left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \rho_a \, c_a \, \frac{\tau_a}{2} \, \frac{\left(T_{bs}' - T_{bs}\right)}{\Delta t} . \qquad (50)$$

Let

$$\beta_a \quad \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad ;$$

rearrange; and solve for T'_{bs}

$$T'_{bs} = T_{bs} + 2\beta_a \left[\frac{\left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{\left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}} \right] \left(T_{bs-1} - T_{bs}\right)$$

$$- \frac{q_{net_i}\,(2\Delta t)}{\rho_a\, c_a\, \tau_a} \left[\frac{\left(R - \sum \tau_{bs}\right)^2}{\left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}} \right] . \qquad (51)$$

b. Thick-Thick Material

At the interface between material "A" and "B" (T_n, Figure 12), the energy balance is

$$\underset{n-1 \to n}{q_{cond}}\, A_1 + \underset{n+1 \to n}{q_{cond}}\, A_2 = \underset{n}{q_{stored}}\, A_3 \qquad (52)$$

where

$$\underset{n-1 \to n}{q_{cond}}\, A_1 = \frac{k_a}{\tau_a}\left(T_{n-1} - T_n\right)\left[\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \phi$$

$$\underset{n+1 \to n}{q_{cond}}\, A_2 = \frac{k_b}{\tau_b}\left(T_{n+1} - T_n\right)\left[\left(R - \sum \tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \phi$$

$$\underset{n}{q_{stored}}\, A_3 = \phi \left\{ \left[\left(R - \sum \tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \rho_a\, c_a \frac{\tau_a}{2} + \left[\left(R - \sum \tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right] \rho_b\, c_b \frac{\tau_b}{2} \right\} \frac{T'_n - T_n}{\Delta t}$$

or

$$\phi\left[\left(R - \sum \tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \frac{k_a}{\tau_a}\left(T_{n-1} - T_n\right) + \phi\left[\left(R - \sum \tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \frac{k_b}{\tau_b}\left(T_{n+1} - T_n\right)$$

$$\phi\left\{\left[\left(R - \sum \tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \rho_a\, c_a \frac{\tau_a}{2} + \left[\left(R - \sum \tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right] \rho_b\, c_b \frac{\tau_b}{2}\right\} \frac{\left(T'_n - T_n\right)}{\Delta t} . \qquad (53)$$

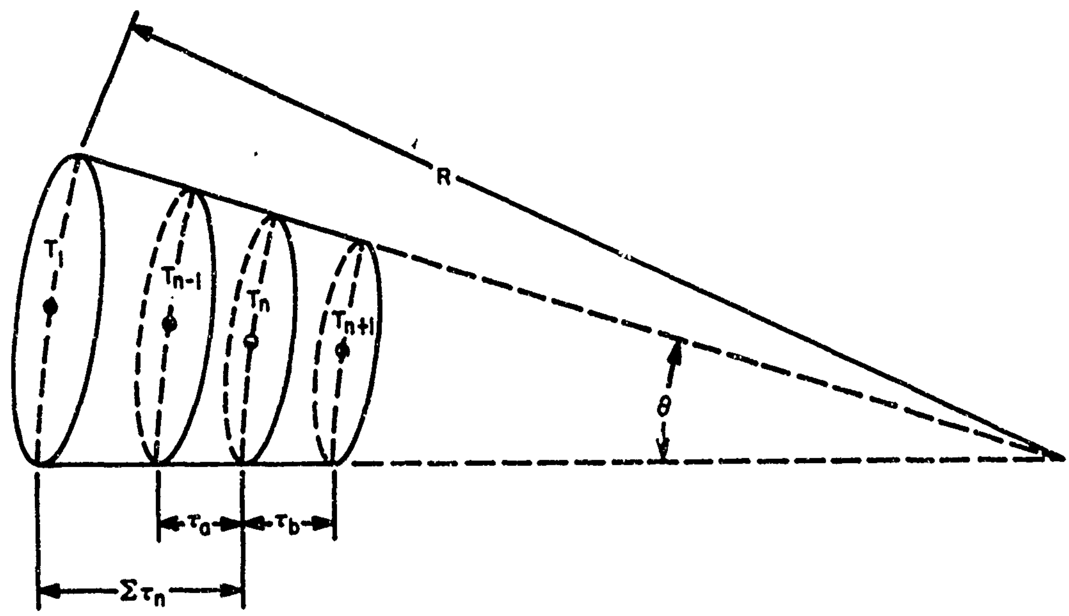


Figure 12.

Rearrange and solve for T_n'

$$T_n' = T_n + \frac{k_a\,\Delta t\left[\left(R - \sum\tau_n + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right]\left(T_{n-1} - T_n\right)}{\tau_a\left\{\left[\left(R - \sum\tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]\rho_a c_a \frac{\tau_a}{2} + \left[\left(R - \sum\tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\rho_b c_b \frac{\tau_b}{2}\right\}}$$

$$+ \frac{k_b\,\Delta t\left[\left(R - \sum\tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]\left(T_{n+1} - T_n\right)}{\tau_b\left\{\left[\left(R - \sum\tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]\rho_a c_a \frac{\tau_a}{2} + \left[\left(R - \sum\tau_n - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\rho_b c_b \frac{\tau_b}{2}\right\}} \quad (54)$$

c. Thin-Thick Material

At the exposed surface (Figure 13) the energy balance for the thermally thin-thermally thick interface (T_1') is

$$q_{net_o} A_1 + q_{\underset{2 \to 1}{cond}} A_2 = q_{\underset{1}{stored}} A_3 \quad (55)$$

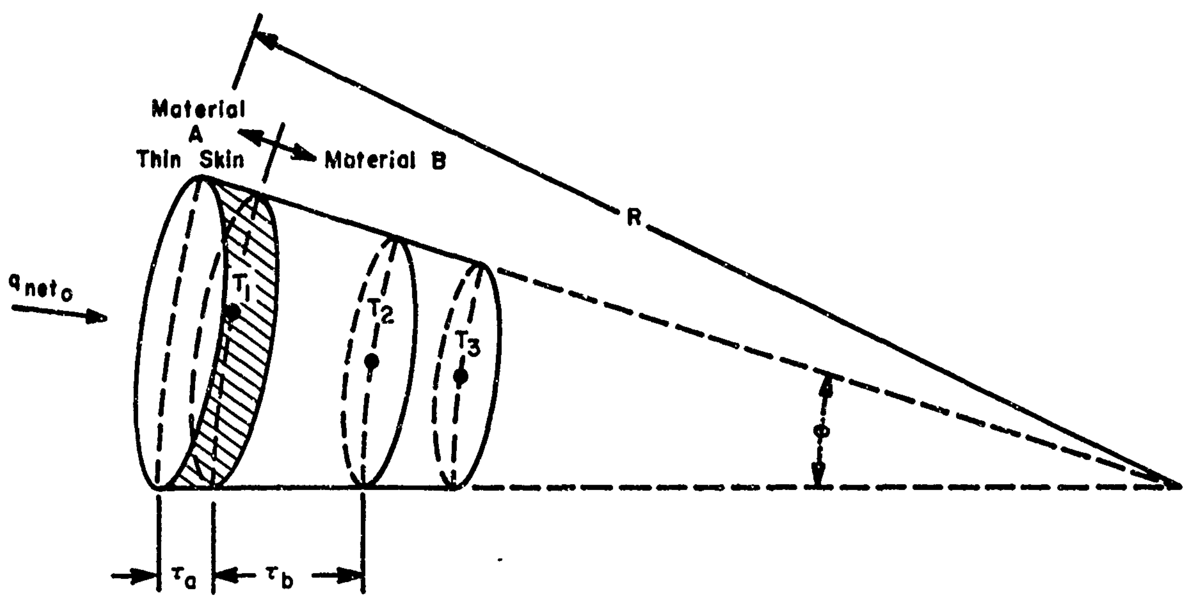


Figure 13.

where

$$q_{net_o} A_1 = q_{net_o} R^2 \phi$$

$$\underset{2 \to 1}{q_{cond}} A_2 = \frac{k_b}{\tau_b} \left(T_2 - T_1\right) \left[\left(R - \tau_a - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \phi$$

$$\underset{1}{q_{stored}} A_3 = \left\{\rho_a c_a \tau_a \left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] + \rho_b c_b \frac{\tau_b}{2} \left[\left(R - \tau_a - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\} \frac{T_1' - T_1}{\Delta t} \phi ,$$

or

$$\phi R^2 q_{net_o} + \phi \frac{k_b}{\tau_b} \left[\left(R - \tau_a - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \left(T_2 - T_1\right) = \phi \left\{\rho_a c_a \tau_a \left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] + \rho_b c_b \frac{\tau_b}{2} \left[\left(R - \tau_a - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\} \frac{T_1' - T_1}{\Delta t} . \quad (56)$$

Rearrange and solve for T_1'

$$T_1' = T_1 + \ldots \frac{q_{net_0}\,\Delta t\, R^2}{\left\{\rho_a c_a \tau_a\left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] + \rho_b c_b \frac{\tau_b}{2}\left[\left(R - \tau_a - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\}} \cdots$$

$$+ \frac{k_b\left[\left(R - \tau_a - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]\left(T_2 - T_1\right)}{\tau_b\left\{\rho_a c_a \tau_a\left[\left(R - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] + \rho_b c_b \frac{\tau_b}{2}\left[\left(R - \tau_a - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\right\}} \qquad (57)$$

d. <u>Thick-Thin Material</u>

The energy balance at the backside surface (Figure 14), T_{bs}, may be written as

$$\underset{bs-1\rightarrow bs}{q_{cond}} A_1 - q_{net_i} A_2 = \underset{bs}{q_{stored}} A_3 \qquad (58)$$

where

$$\underset{bs-1\rightarrow bs}{q_{cond}} A_1 = \frac{k_a}{\tau_a}\left(T_{bs-1} - T_{bs}\right)\left[\left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \phi$$

$$q_{net_i} A_2 = q_{net_i}\left(R - \sum \tau_{bs} - \tau_b\right)^2 \phi$$

$$\underset{bs}{q_{stored}} A_3 = \left\{\rho_a c_a \frac{\tau_a}{2}\left[\left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]\right.$$

$$\left. + \rho_b c_b \tau_b\left[\left(R - \sum \tau_{bs} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]\right\} \frac{\left(T_{bs}' - T_{bs}\right)}{\Delta t} \phi$$

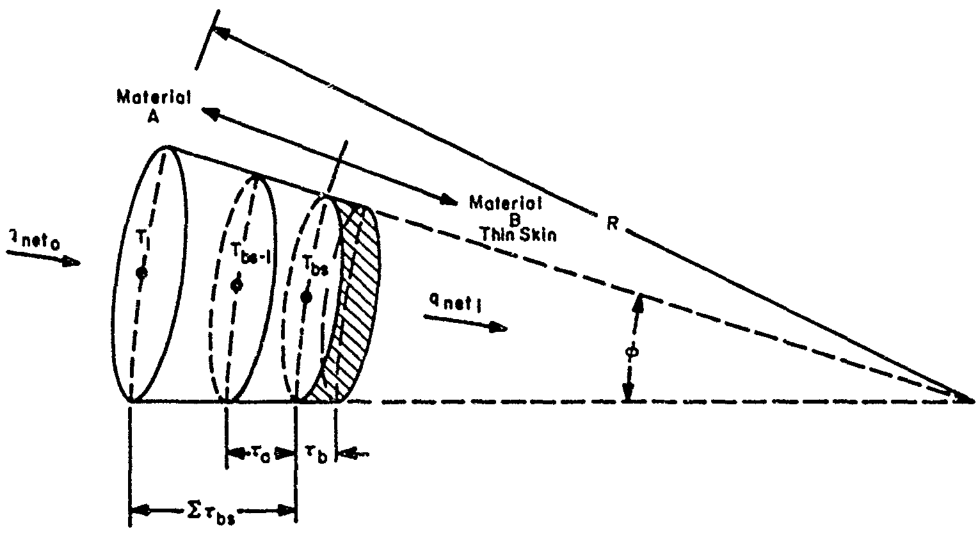


Figure 14.

or

$$\phi \frac{k_a}{\tau_a} \left| \left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12} \right| \left(T_{bs-1} - T_{bs}\right) - \phi\, q_{net_i} \left(R - \sum \tau_{bs} - \tau_b\right)^2$$

$$= \phi \left\{ \rho_a c_a \frac{\tau_a}{2} \left| \left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48} \right| \right.$$

$$\left. + \rho_b c_b \tau_b \left| \left(R - \sum \tau_{bs} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12} \right| \right\} \frac{\left(T'_{bs} - T_{bs}\right)}{\Delta t} . \tag{59}$$

Rearrange and solve for T'_{bs}

$$T'_{bs} = T_{bs} + \frac{k_a \Delta t \left| \left(R - \sum \tau_{bs} + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12} \right| \left(T_{bs-1} - T_{bs}\right)}{\tau_a \left\{ \rho_a c_a \frac{\tau_a}{2} \left| \left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48} \right| + \rho_b c_b \tau_b \left| \left(R - \sum \tau_{bs} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12} \right| \right\}}$$

$$- \frac{q_{net_i} \Delta t \left(R - \sum \tau_{bs} - \tau_{bs}\right)^2}{\rho_a c_a \frac{\tau_a}{2} \left| \left(R - \sum \tau_{bs} + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48} \right| + \rho_b c_b \tau_b \left| \left(R - \sum \tau_{bs} - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12} \right|} . \tag{60}$$

e. <u>Thick-Thin-Thick Material</u>

At the interface between "A" and "C" (T_n, Figure 15), the energy balance for the thermally thin material "B" is

$$\underset{n-1 \to n}{q_{cond}} A_1 + \underset{n+1 \to n}{q_{cond}} A_2 = \underset{n}{q_{stored}} A_3 \tag{61}$$

where

$$\underset{n-1 \to n}{q_{cond}} A_1 = \frac{k_a}{\tau_a} \left(T_{n-1} - T_n\right) \left| \left(R - \sum \tau_n - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12} \right| \phi$$

$$\underset{n+1 \to n}{q_{cond}} A_2 = \frac{k_c}{\tau_c} \left(T_{n+1} - T_n\right) \left| \left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{2}\right)^2 + \frac{\tau_c^2}{12} \right| \phi$$

$$\underset{n}{q_{stored}} A_3 = \left\{ \rho_a c_a \frac{\tau_a}{2} \left| \left(R - \sum \tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48} \right| \right.$$

$$+ \rho_b c_b \tau_b \left| \left(R - \sum \tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12} \right| + \rho_c c_c \frac{\tau_c}{2} \left| \left(R - \sum \tau_n \right. \right.$$

$$\left. \left. \left. - \tau_b - \frac{\tau_c}{4}\right)^2 + \frac{\tau_c^2}{48} \right| \right\} \frac{\left(T'_n - T_n\right)}{\Delta t}$$

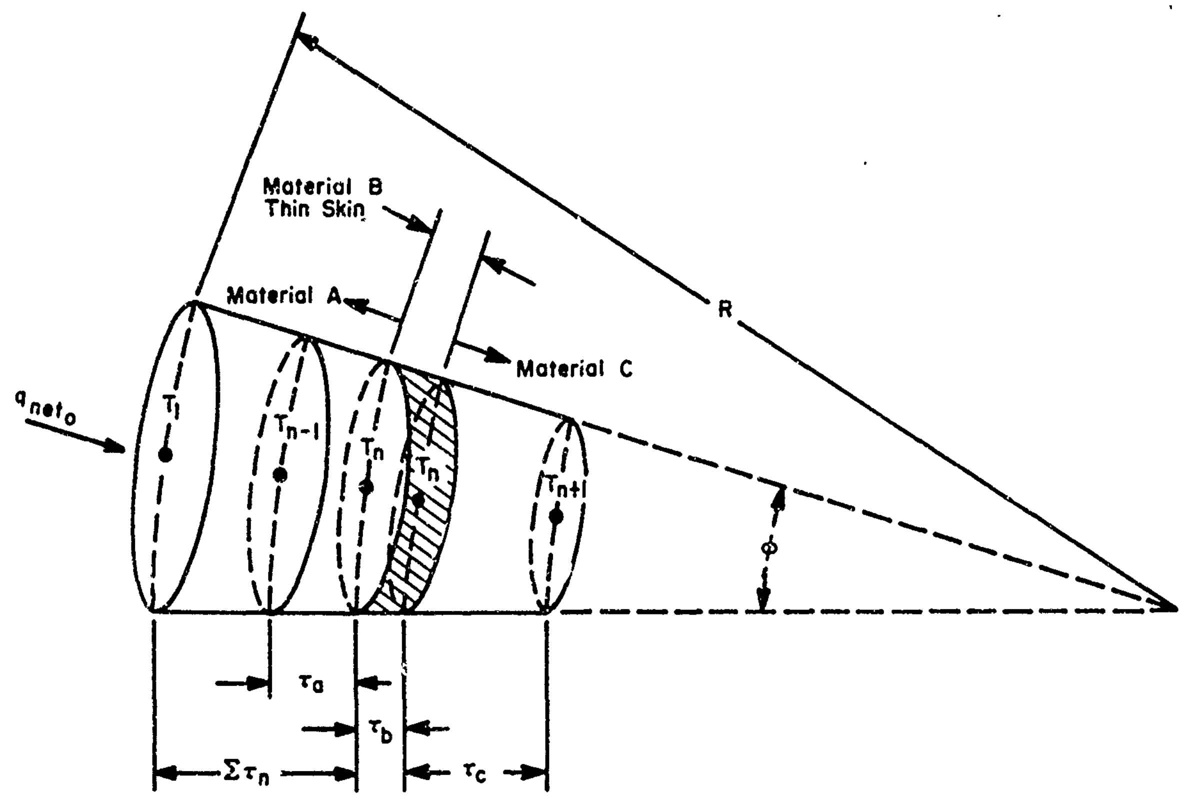


Figure 15.

or

$$\phi \frac{k_a}{\tau_a}\left[\left(R-\sum\tau_n+\frac{\tau_a}{2}\right)^2+\frac{\tau_a^2}{12}\right]\left(T_{n-1}-T_n\right)+\phi\frac{k_c}{\tau_c}\left[\left(R-\sum\tau_n-\tau_b-\frac{\tau_c}{2}\right)^2+\frac{\tau_c^2}{12}\right]\left(T_{n+1}-T_n\right) = \phi\left\{\rho_a c_a \frac{\tau_a}{2}\left[\left(R-\sum\tau_n+\frac{\tau_a}{4}\right)^2+\frac{\tau_a^2}{48}\right] + \rho_b c_b \tau_b\left[\left(R-\sum\tau_n-\frac{\tau_b}{2}\right)^2+\frac{\tau_b^2}{12}\right]+\rho_c c_c\frac{\tau_c}{2}\left[\left(R-\sum\tau_n-\tau_b-\frac{\tau_c}{4}\right)^2+\frac{\tau_c^2}{48}\right]\right\}\frac{\left(T_n'-T_n\right)}{\Delta t} . \qquad (62)$$

Rearrange and solve for T_n'

$$T_n' = T_n + \frac{k_a \Delta t \left[\left(R - \sum \tau_n - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right]\left(T_{n-1} - T_n\right)}{\tau_a\left\{\rho_a c_a \frac{\tau_a}{2}\left[\left(R - \sum \tau_n - \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] + \rho_b c_b \tau_b \left[\left(R - \sum \tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] + \rho_c c_c \frac{\tau_c}{2}\left[\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{4}\right)^2 + \frac{\tau_c^2}{48}\right]\right\}}$$

$$+ \frac{k_c \Delta t \left[\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{2}\right)^2 + \frac{\tau_c^2}{12}\right]\left(T_{n+1} - T_n\right)}{\tau_c\left\{\rho_a c_a \frac{\tau_a}{2}\left[\left(R - \sum \tau_n + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] + \rho_b c_b \tau_b \left[\left(R - \sum \tau_n - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] + \rho_c c_c \frac{\tau_c}{2}\left[\left(R - \sum \tau_n - \tau_b - \frac{\tau_c}{4}\right)^2 + \frac{\tau_c^2}{48}\right]\right\}} \tag{63}$$

Section III. HEAT CONDUCTION DURING ABLATION

To calculate the heat transfer in a material when the exposed surface of the material is heated, an energy balance must be performed. For this analysis the energy is considered to be either radiated away from the exposed surface, conducted into the cooler interior of the structure, or stored in the material near the exposed surface. If sufficient energy is stored at the surface, the surface temperature will eventually reach a critical value. This value is usually known as an ablating, melting, or subliming temperature (T_m). In this report it is assumed that the ablating temperature T_m is known or calculable, and this temperature remains constant while ablation is in process. Another basic parameter required once the exposed surface has reached the ablation temperature, T_m, is the recession rate (ablation rate) or the rate of material removal. It is assumed that the ablation rate ($\dot{a}$) is known or can be calculated for any given increment of time, but may change as a function of time.

It is also assumed that once the exposed surface reaches the melt temperature (T_m), the recession rate governs the amount of material removed. The material properties of specific heat (C) and thermal conductivity (k) may all be a function of temperature.* The pyrolysis of the material leaving the heated surface of the slab has been left out intentionally because of the complexity of the problem. However, this parameter can be included in the heat balance, if desired.

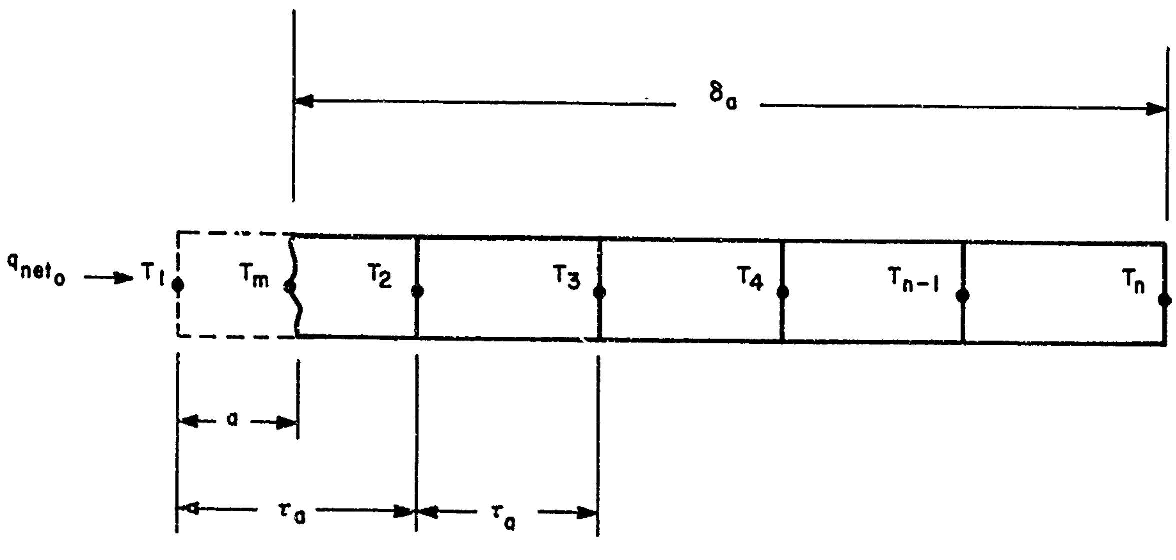
Figure 16 shows the temperature grid arrangement for a slab undergoing surface recession. Nodal points T_2 through T_n do not change or shift positions while ablation is occurring between the original nodal points T_1 and T_2. When the ablation front or receding front reaches or passes T_2, the T_2 nodal point temperature assumes the value of T_m, and T_3' is calculated in the same manner as was T_2' when the receding surface was between nodal points T_1 and T_2. Once the ablation front reaches a nodal point, the ablation distance "a" is reduced by the value of τ_a $(0 \leq a \leq \tau_a)$, and the process starts over again.

Let us first assume that T_1 and T_1' take on a constant value T_m for all times during ablation. The primed or future values for all T's (T_3' through T_n' in Figure 16) can be obtained by ordinary forward finite-difference methods since each of these nodal points is a full τ_a

*Temperature dependent approximations for specific heat and thermal conductivity.

from the two adjacent nodal points. Thus only the future temperature at one nodal point (T_2') is not defined. Examination of the geometry for the ablation case shown in Figure 16 in light of the pure conduction condition described in Figure 1 reveals that two basic conditions must be considered for the ablation-conduction case. One case is for $0 \le a \le \frac{\tau_a}{2}$, and the other is for $\frac{\tau_a}{2} < a \le \tau_a$.

The energy balance for the T_2' nodal point using forward finite-difference approximations for a slab is as follows when "a" is equal to or less than $\frac{\tau_a}{2}$.



Note: $T_1 = T_1' = T_m$ during ablation.

Figure 16.

Recession condition $0 \le a \le \frac{\tau_a}{2}$

$$q_{in\,1\to 2} A - q_{out\,2\to 3} A = q_{stored\,2} A \tag{64}$$

where A is unit area, or

$$\frac{k_a}{\tau_a - a}\left(T_m - T_2\right) + \frac{k_a}{\tau_a}\left(T_3 - T_2\right) = \rho_a\, c_a\, \tau_a \frac{\left(T_2' - T_2\right)}{\Delta t}. \tag{65}$$

Let

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad ;$$

rearrange; and solve for T_2'

$$T_2' = T_2 + \beta_a \left(\frac{\tau_a}{\tau_a - a}\right)\left(T_m - T_2\right) + \beta_a \left(T_3 - T_2\right) \quad . \tag{66}$$

This equation is valid for $0 \le a \le \frac{\tau_a}{2}$. When "a" = 0, Equation (66) reduces to an interior node equation having the form of Equation (6) for the pure conduction condition with $T_m = T_1$. When "a" = $\frac{\tau_a}{2}$, Equation (66) is unstable for $\beta > 1/3$.

For the case where $\frac{\tau_a}{2} < a \le \tau_a$, the reduced storage associated with nodal point 2 must be considered. The energy balance for the T_2' nodal point using forward finite differences becomes

$$\underset{1 \rightarrow 2}{q_{in}} A - \underset{2 \rightarrow 3}{q_{out}} A = \underset{2}{q_{stored}} A \tag{67}$$

where A is unit area, or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2\right) + \frac{k_a}{\tau_a} \left(T_3 - T_2\right) = \rho_a \, c_a \left(\frac{3\tau_a}{2} - a\right) \frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{68}$$

Let

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad ;$$

rearrange; and solve for T_2'

$$T_2' = T_2 + \beta_a \frac{\tau_a^2 \left(T_m - T_2\right)}{\left(\tau_a - a\right)\left(\frac{3\tau_a}{2} - a\right)} + \beta_a \frac{\tau_a}{\left(\frac{3\tau_a}{2} - a\right)} \left(T_3 - T_2\right) \quad . \tag{69}$$

Equation (69) is discontinuous or unstable at "a" = τ_a, regardless of the value of β. This instability would bring about problems each time "a"$\rightarrow \tau_a$ and nodal points are removed.

To eliminate the stability problems associated with Equations (66) and (69), an investigation was made of a backward finite-difference approximation for T_2' when ablation is in process. The backward finite-difference approximation is accomplished by priming all temperature values to the left of the equal sign in Equations (65) and (68). This is possible since the surface temperature, T_m, is known, and all other temperatures beyond T_2 are easily obtained by standard forward-difference techniques. Two resulting equations in terms of T_2' are obtained in relation to the position of the receding surface.

Recession condition $0 \le a \le \frac{\tau_a}{2}$.

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2'\right) + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) = \rho_a c_a \tau_a \frac{\left(T_2' - T_2\right)}{\Delta t}, \quad 0 \le a \le \frac{\tau_a}{2} \tag{70}$$

or letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2}$$

and solving for T_2'

$$T_2' = \frac{\beta_a T_m + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) T_3' + T_2\left(\frac{\tau_a - a}{\tau_a}\right)}{\beta_a + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) + \frac{\tau_a - a}{\tau_a}}, \quad 0 \le a \le \frac{\tau_a}{2} \tag{71}$$

Recession condition $\frac{\tau}{2} < a \le \tau_a$

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2'\right) + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) = \rho_a c_a \left(\frac{3\tau_a}{2} - a\right) \frac{\left(T_2' - T_2\right)}{\Delta t}, \quad \frac{\tau_a}{2} < a \le \tau_a \tag{72}$$

or letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and solving for T_2'

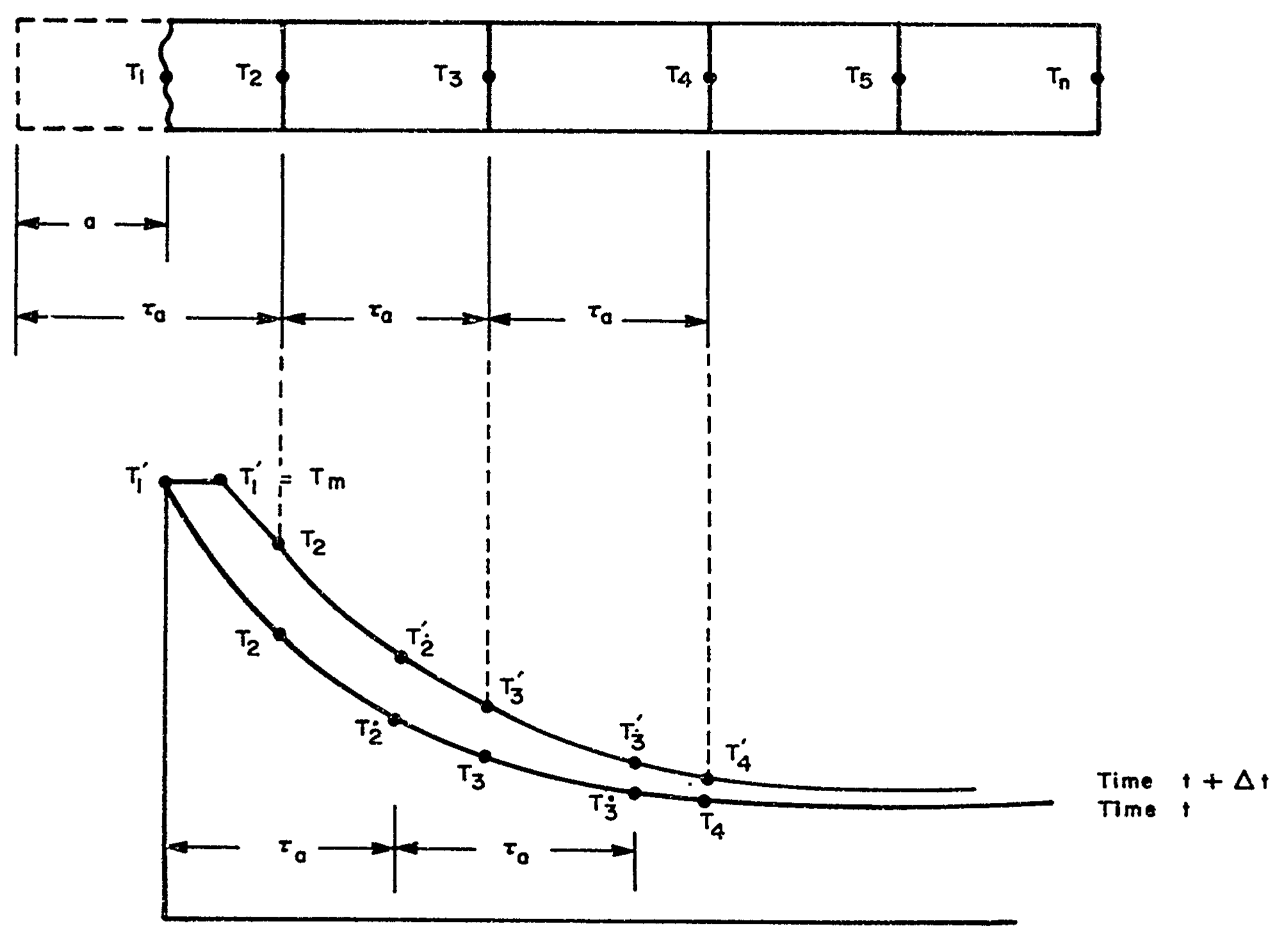
$$T_2' = \frac{\beta_a \, T_m + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) \; T_3' + T_2 \; \frac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}}{\beta_a + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) + \frac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}} ,$$

$$\frac{\tau_a}{2} < a \leq \tau_a \; . \qquad (73)$$

It can readily be seen that Equations (71) and (73) are continuous for all values of "a" ($0 \leq a \leq \tau_a$) and are a marked improvement over Equations (66) and (69).

A mid finite-difference approximation was also investigated for finding T_2'. This mid-difference method is a better approximation to the exact solution than the backward finite-difference approach. However, the additional complexity of the equations and computer storage requirements were considered unwarranted for the additional accuracy gained.

Another ablation-conduction approach that has been used successfully is the "shift" method shown in Figure 17. An interpolation routine is used with known temperatures T_1, T_2, T_3, and T_4 to get $T_{\dot{2}}$ and $T_{\dot{3}}$ located at even increments of τ_a from T_1. The general forward finite-difference interior equation is used to get $T_{\dot{2}}'$, i.e., T_1, $T_{\dot{2}}$, and $T_{\dot{3}}$ are used to get $T_{\dot{2}}'$. Nodal points T_3' through T_n' are calculated using the ordinary forward finite-difference equations and original temperature node locations. With the prime temperatures known (T_1', $T_{\dot{2}}'$, $T_{\dot{3}}'$), it is possible to use a three-point interpolation routine to find T_2' located at a distance of one τ_a in front of nodal point T_3' and τ_a - a distance from the receding surface. This "shift" method can be used accurately until the receding surface becomes closer than two τ_a's from the unheated surface. At this time other special equations must be utilized.



T1
T2
T3
T4
T5
Tn
a
τa
τa
τa
T1'
T1' = Tm
T2
T2'
T3'
T3'
T4'
T2
T2'
T3
T3'
T4
τa
τa
Time t + Δt
Time t

Figure 17.

After consideration of the advantages and disadvantages of the three approaches discussed, it was decided to pursue the backward finite-difference method in deriving the energy balance equations (T_2') during ablation for one-dimensional flat plates and radial conduction in cylinders and spheres.

The finite-difference equations presented in this report for finding the temperature profile near the receding surface are limited to the condition of $0 \leq a \leq \tau_a$. However, as the receding front passes successive originally selected temperature nodes, the equations as derived are applicable if appropriate temperature subscripts are used. For example, when the receding front is between the original location of T_2 and T_3 (Figure 16) on a semi-infinite slab, either Equation (76) or (79) is used to find T_3' by increasing all temperature subscripts by one.

1. Flat Plate

The procedure for calculating heat flow during surface recession is described for all expected material combinations with the general thick equations being a repeat of Equations (70) through (74).

a. <u>General Thick with $\delta_a > \tau_a$ (Figure 16)</u>

(1) <u>$0 \leq a \leq \frac{\tau_a}{2}$</u>. The energy balance for T_2 is

$$\underset{1\to 2}{q_{in}} - \underset{2\to 3}{q_{out}} = \underset{2}{q_{stored}} \tag{74}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)}\left(T_m - T_2'\right) + \frac{k_a}{\tau_a}\left(T_3' - T_2'\right) = \rho_a\, c_a\, \tau_a \frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{75}$$

Letting

$$\beta_a = \frac{k_a\, \Delta t}{\rho_a\, c_a\, \tau_a^2}$$

and solving for T_2',

$$T_2' = \frac{\beta_a T_m + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) T_3' + T_2\left(\frac{\tau_a - a}{\tau_a}\right)}{\beta_a + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) + \frac{\tau_a - a}{\tau_a}} . \tag{76}$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$. The energy balance for T_2 is

$$\underset{1\to 2}{q_{in}} - \underset{2\to 3}{q_{out}} = \underset{2}{q_{stored}} \tag{77}$$

or

$$\frac{k_a}{\tau_a - a}\left(T_m - T_2'\right) + \frac{k_a}{\tau_a}\left(T_3' - T_2'\right) = \rho_a c_a \left(\frac{3\tau_a}{2} - a\right)\frac{\left(T_2' - T_2\right)}{\Delta t} . \tag{78}$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2}$$

and solving for T_2',

$$T_2' = \frac{\beta_a T_m + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) T_3' + T_2 \frac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}}{\beta_a + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right) + \frac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}} . \tag{79}$$

It should be noted in Equation (79) that, as "a" approaches τ_a, T_2' takes on the value of T_m. This is true under actual conditions since the nodal point T_1 is moving, and all the other nodal points are fixed. As the ablation front reaches the original interior nodal points, these nodes take on a temperature value of T_m.

As long as the remaining wall thickness of the material undergoing recession is equal to one or more than one τ, the interior and interface equations presented in Section II are used to solve the heat balances throughout the structure away from the receding surfaces.

b. Special Thick with $\delta_a \leq \tau_a$ (Receding Surface $\leq$ Distance τ_a from Backside, Figure 18)

When the receding surface is less than one incremental τ_a from another material surface or the backside of material "A," special consideration must be made for all boundaries normally experienced.

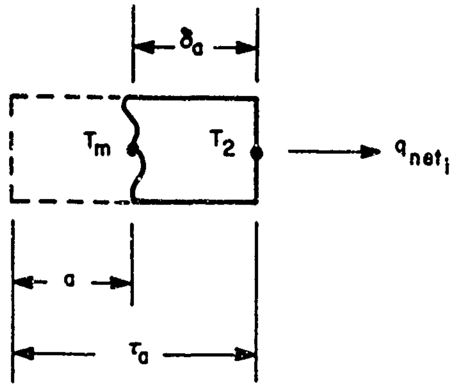


Figure 18.

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_2 is

$$q_{\substack{cond \\ 1 \to 2}} - q_{net_i} = q_{\substack{stored \\ 2}} \tag{80}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2'\right) - q_{net_i} = \rho_a \, c_a \, \frac{\tau_a}{2} \; \frac{\left(T_2' - T_2\right)}{\Delta t} \; . \tag{81}$$

Letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a{}^2}$$

and solving for T_2',

$$T_2' = \frac{2\beta_a \, T_m + \left(\frac{\tau_a - a}{\tau_a}\right) \; T_2 - \frac{2 \, q_{net_i} \, \Delta t}{\rho_a \, c_a \, \tau_a} \; \left(\frac{\tau_a - a}{\tau_a}\right)}{2\beta_a + \frac{\tau_a - a}{\tau_a}} \; . \tag{82}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance for T_2 is

$$q_{\underset{1\to 2}{cond}} - q_{net_i} = q_{\underset{2}{stored}} \tag{83}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) - q_{net_i} = \rho_a\, c_a \left(\frac{\tau_a - a}{2}\right) \frac{T_2' - T_2}{\Delta t} \quad . \tag{84}$$

Letting

$$\beta_a = \frac{k_a\, \Delta t}{\rho_a\, c_a\, \tau_a^2}$$

and solving for T_2',

$$T_2' = \frac{2\beta_a\, T_m + \left(\frac{\tau_a - a}{\tau_a}\right)^2 T_2 - \frac{2\, q_{net_i}}{\rho_a\, c_a\, \tau_a}\, \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{2\beta_a + \left(\frac{\tau_a - a}{\tau_a}\right)^2} \quad . \tag{85}$$

c. <u>Special Thick-Thin with $\delta_a \le \tau_a$ (Figure 19)</u>

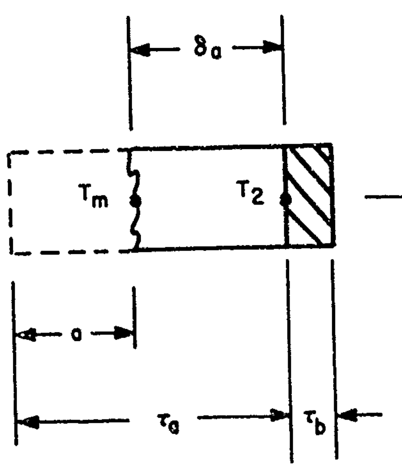


Figure 19.

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$. The energy balance for T_2 is

$$q_{\substack{cond \\ 1 \to 2}} - q_{net_i} = q_{\substack{stored \\ 2}} \tag{86}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2'\right) - q_{net_i} = \left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right) \frac{\left(T_2' - T_2\right)}{\Delta t} \, . \tag{87}$$

Solving for T_2',

$$T_2' = \frac{T_m \dfrac{k_a \Delta t}{\tau_a \left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right]} + T_2 \left(\dfrac{\tau_a - a}{\tau_a}\right) - \dfrac{q_{net_i} \Delta t}{\left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right]} \left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t}{\tau_a \left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right]} + \dfrac{\tau_a - a}{\tau_a}} \, . \tag{88}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance is

$$q_{\substack{cond \\ 1 \to 2}} - q_{net_i} = q_{\substack{stored \\ 2}} \tag{89}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2'\right) - q_{net_i} = \left(\rho_a c_a \frac{\tau_a - a}{2} + \rho_b c_b \tau_b\right) \frac{\left(T_2' - T_2\right)}{\Delta t} \, . \tag{90}$$

Solving for T_2',

$$T_2' = \frac{\dfrac{T_m k_a \Delta t}{\tau_a \left[\rho_a c_a \frac{\tau_a - a}{2} + \rho_b c_b \tau_b\right]} + T_2 \left(\dfrac{\tau_a - a}{\tau_a}\right) - \dfrac{q_{net_i} \Delta t}{\left[\rho_a c_a \frac{\tau_a - a}{2} + \rho_b c_b \tau_b\right]} \left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t}{\tau_a \left[\rho_a c_a \frac{\tau_a - a}{2} + \rho_b c_b \tau_b\right]} + \dfrac{\tau_a - a}{\tau_a}} \, . \tag{91}$$

d. Special Thick-Thick With $\delta_a < \tau_a$ (Figure 20)

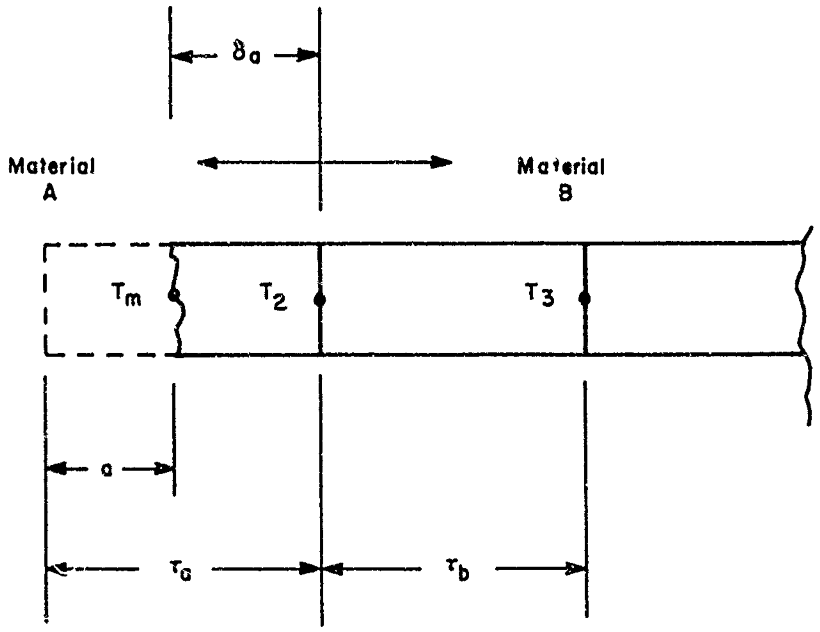


Figure 20.

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance is

$$q_{\underset{1 \to 2}{cond}} + q_{\underset{3 \to 2}{cond}} = q_{\underset{2}{stored}} \tag{92}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2'\right) + \frac{k_b}{\tau_b} \left(T_3' - T_2'\right) = \left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \frac{\tau_b}{2}\right) \frac{\left(T_2' - T_2\right)}{\Delta t} . \tag{93}$$

Rearranging and solving for T_2',

$$T_2' = \frac{T_m \dfrac{k_a \Delta t}{\tau_a \left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \frac{\tau_b}{2}\right]} + T_3' \dfrac{k_b \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b \left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \frac{\tau_b}{2}\right]} + T_2 \left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t}{\tau_a \left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{k_b \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b \left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{\tau_a - a}{\tau_a}} . \tag{94}$$

(2) $\frac{\tau_a}{2} < a \le \tau_a$. The energy balance is

$$q_{\underset{1 \to 2}{cond}} + q_{\underset{3 \to 2}{cond}} \quad q_{\underset{2}{stored}} \tag{95}$$

or

$$\frac{k_a}{\tau_a - a} \left(T_m - T_2'\right) + \frac{k_b}{\tau_b} \left(T_3' - T_2'\right) = \left[\rho_a\, c_a \left(\tau_a - a\right) + \rho_b\, c_b \frac{\tau_b}{2}\right] \frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{96}$$

Solving for T_2',

$$T_2' = \frac{\dfrac{T_m\, k_a\, \Delta t}{\tau_a\left[\rho_a\, c_a\left(\tau_a - a\right) + \rho_b\, c_b \frac{\tau_b}{2}\right]} + T_3' \dfrac{k_b\, \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b\left[\rho_a\, c_a\left(\tau_a - a\right) + \rho_b\, c_b \frac{\tau_b}{2}\right]} + T_2\left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t}{\tau_a\left[\rho_a\, c_a\left(\tau_a - a\right) + \rho_b\, c_b \frac{\tau_b}{2}\right]} + \dfrac{k_b\, \Delta t\left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b\left[\rho_a\, c_a\left(\tau_a - a\right) + \rho_b\, c_b \frac{\tau_b}{2}\right]} + \dfrac{\tau_a - a}{\tau_a}} \quad . \tag{97}$$

Heat flow in the second thick layer is determined from those equations presented in Section II for a nonrecession case.

e. Special Thick-Thin-Thick with $\delta_a \le \tau_a$ (Figure 21).

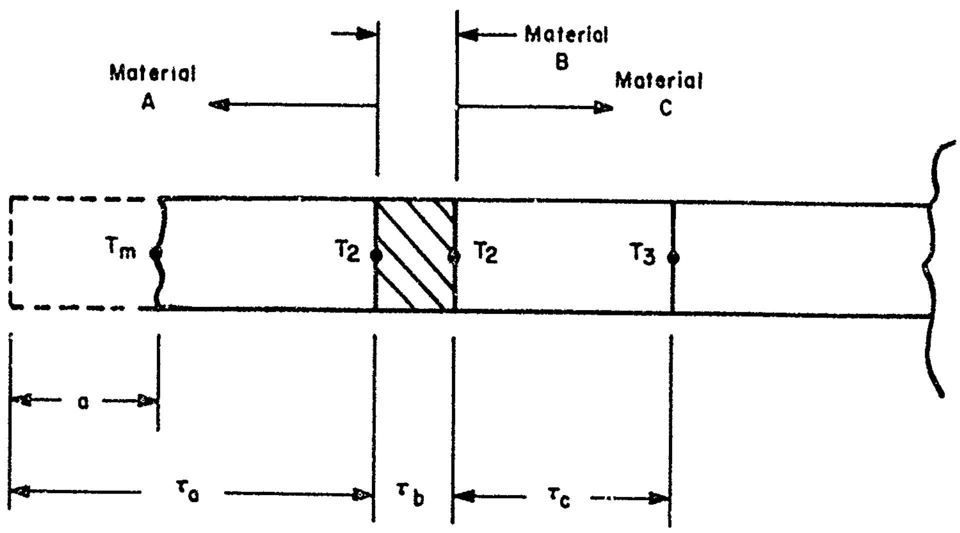


Figure 21.

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$. The energy balance is

$$q_{\substack{cond \\ 1 \to 2}} + q_{\substack{cond \\ 3 \to 2}} = q_{\substack{stored \\ 2}} \tag{98}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) + \frac{k_c}{\tau_c} \left(T_3' - T_2'\right) = \left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right] \frac{(T_2' - T_2)}{\Delta t} \quad . \tag{99}$$

Solving for T_2',

$$T_2' = \frac{\dfrac{T_m k_a \Delta t}{\tau_a\left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + \dfrac{T_3' k_c \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_c\left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + T_2\left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t}{\tau_a\left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + \dfrac{k_c \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_c\left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + \left(\dfrac{\tau_a - a}{\tau_a}\right)} \quad . \tag{100}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance is

$$q_{\substack{cond \\ 1 \to 2}} + q_{\substack{cond \\ 3 \to 2}} = q_{\substack{stored \\ 2}} \tag{101}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) + \frac{k_c}{\tau_c} \left(T_3' - T_2'\right) = \left[\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right] \frac{(T_2' - T_2)}{\Delta t} \quad . \tag{102}$$

Solving for T_2',

$$T_2' = \frac{\dfrac{T_m k_a \Delta t}{\tau_a\left[\rho_a c_a (\tau_a - a) + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + \dfrac{T_3' k_c \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_c\left[\rho_a c_a (\tau_a - a) + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + T_2\left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t}{\tau_a\left[\rho_a c_a (\tau_a - a) + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + \dfrac{k_c \Delta t \, \frac{(\tau_a - a)}{\tau_a}}{\tau_c\left[\rho_a c_a (\tau_a - a) + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right]} + \dfrac{\tau_a - a}{\tau_a}} \quad . \tag{103}$$

Heat flow in the second thick layer is determined from those equations presented in Section II for a nonrecession case.

2. Cylinder Ablation-Conduction

The energy balance for the radial heat flow toward the centerline in a cylinder is basically the same as the flat-plate case, with the exception that average areas or changing areas are considered for conduction and storage. Once the ablation front reaches an interior nodal point, the ablation distance "a" is reduced by the value of τ_a ($0 \le a \le \tau_a$), and the finite-difference calculation process starts over again. In addition, the radius to the original T_2 nodal point is reduced by the value τ_a.

a. General Thick with $\delta_a > \tau_a$ (Figure 22)

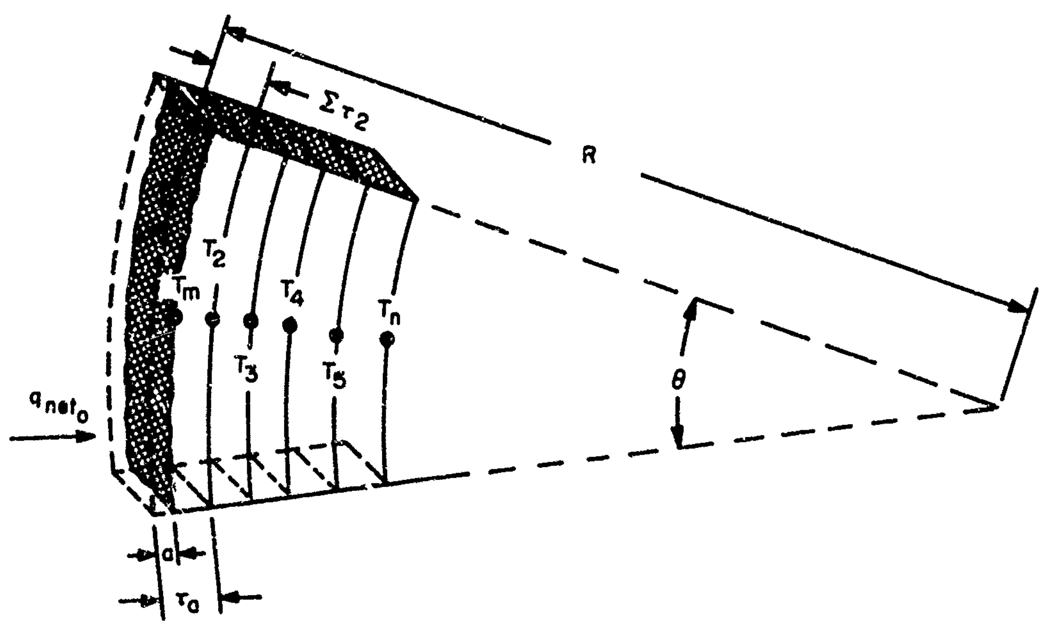


Figure 22.

(1) $0 \le a \le \frac{\tau_a}{2}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} A_{1-2} + q_{\underset{3 \to 2}{cond}} A_{3-2} = q_{\underset{2}{stored}} A_2 \tag{104}$$

or

$$\frac{k_a}{(\tau_a - a)}\left(T_m - T_2'\right)A_{1-2} + \frac{k_a}{\tau_a}\left(T_3' - T_2'\right)A_{3-2}$$

$$= \rho_a c_a \tau_a \frac{(T_2' - T_2)}{\Delta t} A_2 \quad . \qquad (105)$$

where $A_{1-2} = 0L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$

$A_{3-2} = 0L\left(R - \sum \tau_2 - \frac{\tau_a}{2}\right)$

$A_2 = 0L\left(R - \sum \tau_2\right)$.

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad , \quad Z = R - \sum \tau_2 \; ,$$

and solving for T_2' ,

$$T_2' = \frac{T_m \beta_a \left(\frac{Z + \frac{\tau_a - a}{2}}{Z}\right) + T_3' \beta_a \left(\frac{Z - \frac{\tau_a}{2}}{Z}\right)\left(\frac{\tau_a - a}{\tau_a}\right) + T_2\left(\frac{\tau_a - a}{\tau_a}\right)}{\beta_a\left(\frac{Z + \frac{\tau_a - a}{2}}{Z}\right) + \beta_a\left(\frac{Z - \frac{\tau_a}{2}}{Z}\right)\left(\frac{\tau_a - a}{\tau_a}\right) + \frac{\tau_a - a}{\tau_a}} \quad . \qquad (106)$$

(2) $\frac{\tau_a}{2} < a \le \tau_a$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} A_{1-2} + q_{\underset{3 \to 2}{cond}} A_{3-2} = q_{\underset{2}{stored}} A_2 \qquad (107)$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) \underset{3-2}{A}$$

$$- \rho_a c_a \left(\frac{3\tau_a}{2} - a\right) \frac{\left(T_2' - T_2\right)}{\Delta t} A_2 \qquad (108)$$

where

$$\underset{1-2}{A} = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$\underset{3-2}{A} = \theta L \left(R - \sum \tau_2 - \frac{\tau_a}{2}\right)$$

$$A_2 = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - 2a}{4}\right) \quad .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad , \quad Z = R - \sum \tau_2 \, ,$$

and solving for T_2',

$$T_2' = \frac{T_m \beta_a \left(\dfrac{Z + \dfrac{\tau_a - a}{2}}{Z + \dfrac{\tau_a - 2a}{4}}\right) + T_3' \beta_a \left(\dfrac{\tau_a - a}{\tau_a}\right) \left(\dfrac{Z - \dfrac{\tau_a}{2}}{Z + \dfrac{\tau_a - 2a}{4}}\right) + T_2 \dfrac{\left(\dfrac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}}{\beta_a \left(\dfrac{Z + \dfrac{\tau_a - a}{2}}{Z + \dfrac{\tau_a - 2a}{4}}\right) + \beta_a \left(\dfrac{\tau_a - a}{\tau_a}\right) \left(\dfrac{Z - \dfrac{\tau_a}{2}}{Z + \dfrac{\tau_a - 2a}{4}}\right) + \dfrac{\left(\dfrac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}} \quad . \qquad (109)$$

As long as the remaining wall thickness of the material undergoing recession is equal to one or more thar one τ, the interior and interface equations presented in Section II are used to solve the heat balances throughout the structure away from the receding surface.

b. <u>Special Thick with $\delta_a \le \tau_a$ (Receding Surface $\le \tau_a$ from Backside) (Figure 23)</u>

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$. The energy balance for T_2 is

$$\underset{1 \to 2}{q_{cond}} \underset{1-2}{A} - q_{net_1} A_i \quad \underset{2}{q_{stored}} A_2 \qquad (110)$$

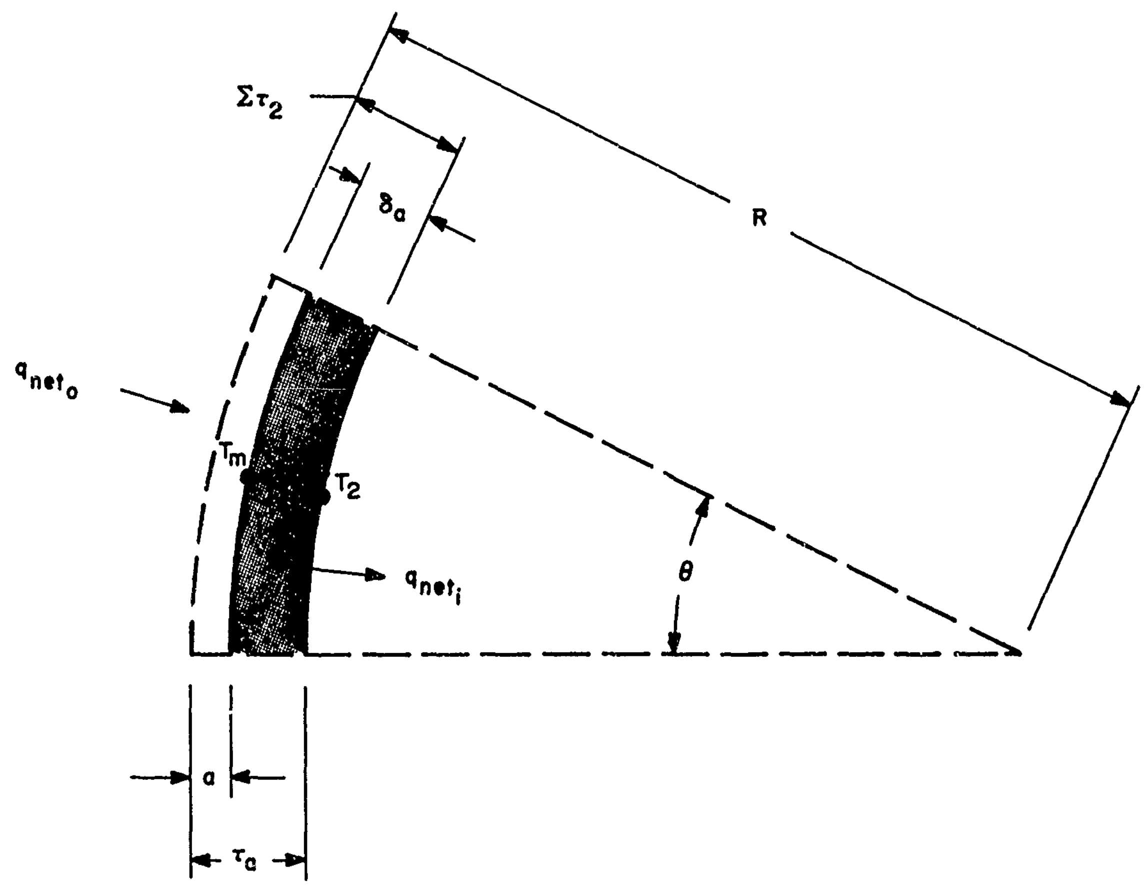


Figure 23.

or

$$\frac{k_a}{\tau_a - a}\left(T_m - T_2'\right) A_{1-2} - q_{net_i} A_i$$

$$= \rho_a c_a \frac{\tau_a}{2} \frac{\left(T_2' - T_2\right)}{\Delta t} A_2 \qquad (111)$$

where

$$A_{1-2} = \theta L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_i = \theta L\left(R - \sum \tau_2\right)$$

$$A_2 = \theta L\left(R - \sum \tau_2 + \frac{\tau_a}{4}\right) \quad .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad , \quad Z = R - \sum \tau_2 \ ,$$

and solving for T_2',

$$T_2' = \frac{T_m 2\beta_a \left(\frac{Z + \frac{\tau_a - a}{2}}{Z + \frac{\tau_a}{4}}\right) - \frac{2 q_{net_i} \Delta t Z}{\left(Z + \frac{\tau_a}{4}\right) \rho_a c_a \tau_a} \left(\frac{\tau_a - a}{\tau_a}\right) + T_2 \left(\frac{\tau_a - a}{\tau_a}\right)}{2\beta_a \left(\frac{Z + \frac{\tau_a - a}{2}}{Z + \frac{\tau_a}{4}}\right) + \frac{\tau_a - a}{\tau_a}} \quad . \tag{112}$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} \underset{1-2}{A} - q_{net_i} A_i = q_{\underset{2}{stored}} A_2 \tag{113}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} - q_{net_i} A_i$$

$$= \rho_a c_a \left(\tau_a - a\right) \frac{\left(T_2' - T_2\right)}{\Delta t} A_2 \quad . \tag{114}$$

where

$$\underset{1-2}{A} = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_i = \theta L \left(R - \sum \tau_2\right)$$

$$A_2 = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right) \quad .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad , \quad Z = R - \sum \tau_2 \quad ,$$

and solving for T_2',

$$T_2' = \frac{T_m \beta_a - \dfrac{q\ net_i\ \Delta t}{\rho_a\ c_a\ \tau_a}\left(\dfrac{\tau_a - a}{\tau_a}\right)\left(\dfrac{Z}{Z + \dfrac{\tau_a - a}{2}}\right) + T_2\left(\dfrac{\tau_a - a}{\tau_a}\right)^2}{\beta_a + \left(\dfrac{\tau_a - a}{\tau_a}\right)^2} \qquad (115)$$

c. Special Thick-Thin with $\delta_a \le \tau_a$ (Figure 24)

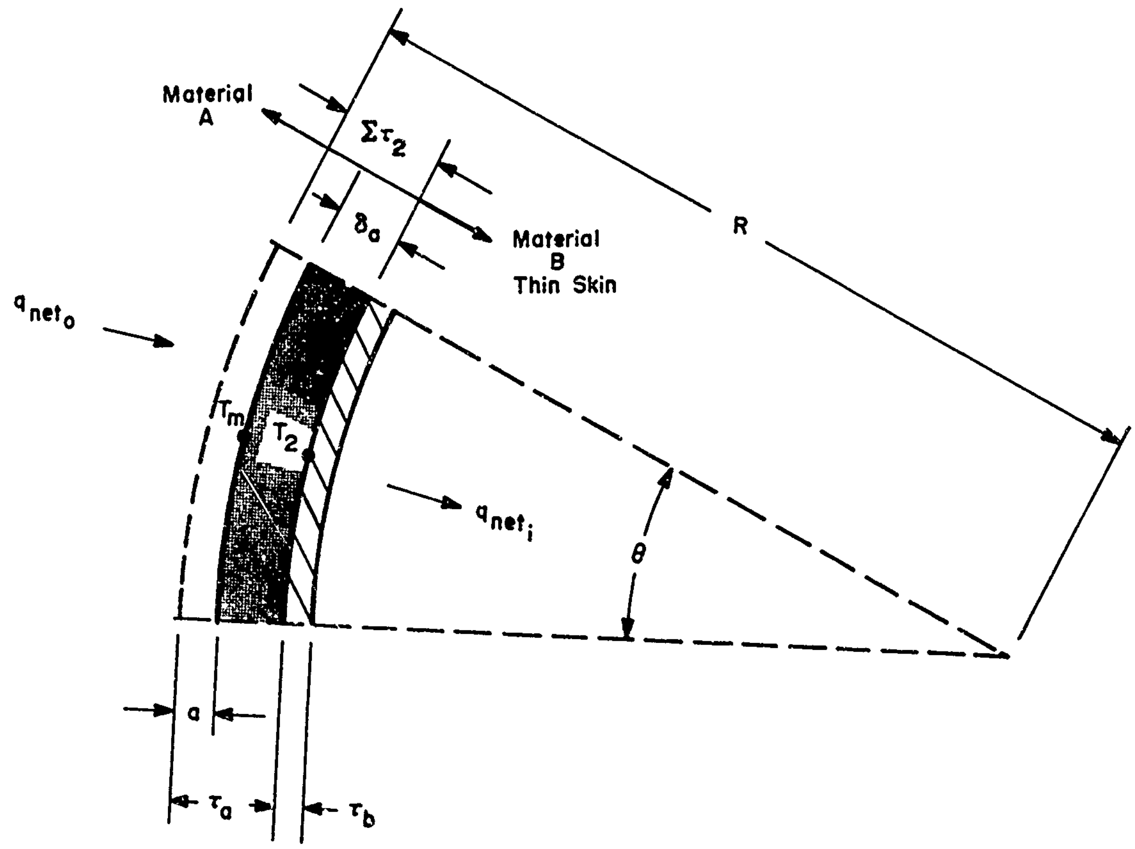


Figure 24.

(1) $0 \le a \le \dfrac{\tau_a}{2}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}}\ A_{1-2} - q_{net_i}\ A_i = q_{\underset{2}{stored}}\ (A_3 + A_4) \qquad (116)$$

$$\frac{k_a}{(\tau_a - a)}\left(T_m - T_2'\right) A_{1-2} - q_{net_i} A_i = \left[\rho_a c_a \frac{\tau_a}{2} A_3 + \rho_b c_b \tau_b A_4\right] \frac{(T_2' - T_2)}{\Delta t} \tag{117}$$

where

$$A_{1-2} = 0L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_i = 0L\left(R - \sum \tau_2 - \tau_b\right)$$

$$A_3 = 0L\left(R - \sum \tau_2 + \frac{\tau_a}{4}\right)$$

$$A_4 = 0L\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right) .$$

Letting $Z = R - \sum \tau_2$ and solving for T_2',

$$T_2' = \frac{\dfrac{T_m k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a \left[\left(Z + \frac{\tau_a}{4}\right)\rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{2}\right)\rho_b c_b \tau_b\right]} - \dfrac{q_{net_i} \Delta t \left(Z - \tau_b\right)\left(\frac{\tau_a - a}{\tau_a}\right)}{\left[\left(Z + \frac{\tau_a}{4}\right)\rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{2}\right)\rho_b c_b \tau_b\right]} + T_2\left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a\left[\left(Z + \frac{\tau_a}{4}\right)\rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{2}\right)\rho_b c_b \tau_b\right]} + \dfrac{\tau_a - a}{\tau_a}} . \tag{118}$$

(2) $\frac{\tau_a}{2} < a \le \tau_a$. The energy balance for T_2 is

$$q_{cond_{1\to 2}} A_{1-2} - q_{net_i} A_i = q_{stored_2}\left(A_3 + A_4\right) \tag{119}$$

$$\frac{k_a}{(\tau_a - a)}\left(T_m - T_2'\right) A_{1-2} - q_{net_i} A_i = \left[\rho_a c_a \left(\tau_a - a\right) A_3 + \rho_b c_b \tau_b A_4\right] \frac{(T_2' - T_2)}{\Delta t} \tag{120}$$

where

$$A_{1-2} = 0L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_i = 0L\left(R - \sum \tau_2 - \tau_b\right)$$

$$A_3 = 0L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_4 = 0L\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right) .$$

Letting $Z = R - \sum \tau_2$ and solving for T_2',

[illegible] (121)

d. <u>Special Thick-Thick with $\delta_a \leq \tau_a$ (Figure 25)</u>

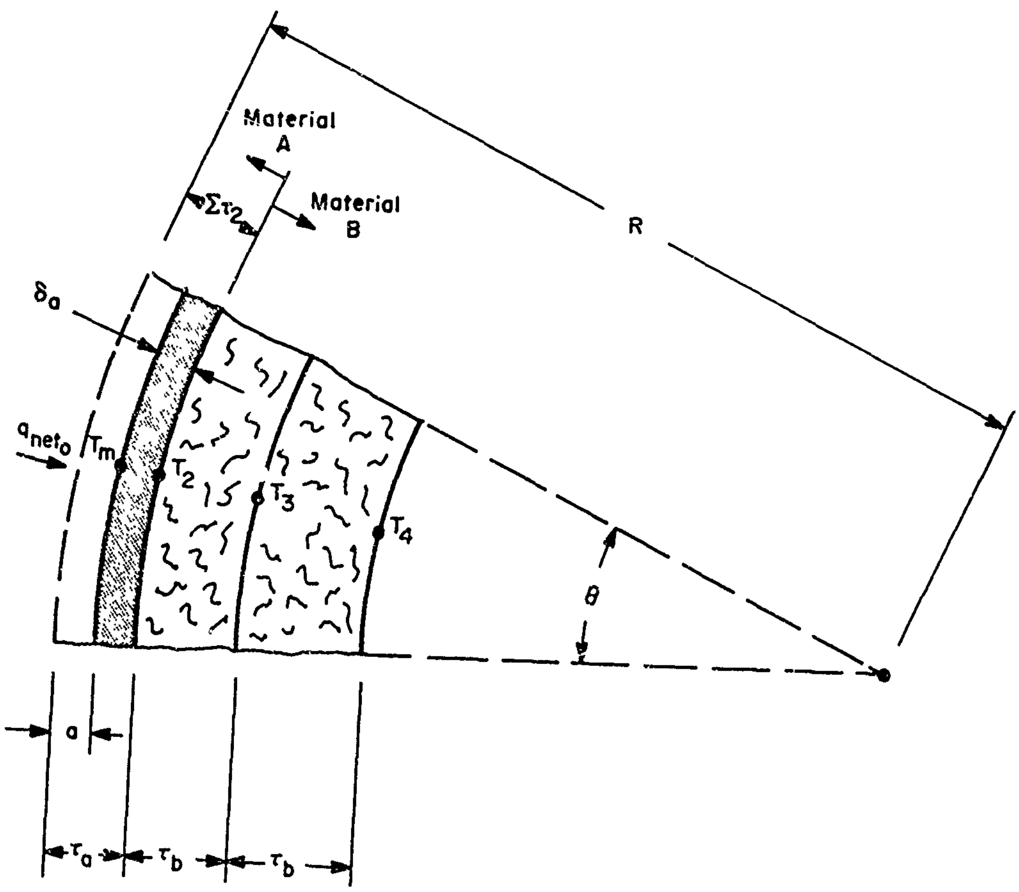


Figure 25.

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} A_{1-2} + q_{\underset{3 \to 2}{cond}} A_{3-2} \quad q_{\underset{2}{stored}} \left(A_3 + A_4\right) \tag{122}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_b}{\tau_b} \left(T_3' - T_2'\right) \underset{3-2}{A} \left[\rho_a c_a \frac{\tau_a}{2} A_3 + \rho_b c_b \frac{\tau_b}{2} A_4\right] \frac{(T_2' - T_2)}{\Delta t} \tag{123}$$

where

$$\underset{1-2}{A} = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$\underset{3-2}{A} = \theta L \left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)$$

$$A_3 = \theta L \left(R - \sum \tau_2 + \frac{\tau_a}{4}\right)$$

$$A_4 = \theta L \left(R - \sum \tau_2 - \frac{\tau_b}{4}\right) .$$

Letting $Z = R - \sum \tau_2$ and solving for T_2',

$$T_2' = \frac{\dfrac{T_m k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a \left[\left(Z + \frac{\tau_a}{4}\right) \rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{4}\right) \rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{T_3' k_b \Delta t \left(Z - \frac{\tau_b}{2}\right)\left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b \left[\left(Z + \frac{\tau_a}{4}\right) \rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{4}\right) \rho_b c_b \frac{\tau_b}{2}\right]} + T_2 \left(\frac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a \left[\left(Z + \frac{\tau_a}{4}\right) \rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{4}\right) \rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{k_b \Delta t \left(Z - \frac{\tau_b}{2}\right)\left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b \left[\left(Z + \frac{\tau_a}{4}\right) \rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{4}\right) \rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{\tau_a - a}{\tau_a}} . \tag{124}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance for T_2 is

$$\underset{1 \to 2}{q_{cond}} \underset{1-2}{A} + \underset{3 \to 2}{q_{cond}} \underset{3-2}{A} = \underset{2}{q_{stored}} \left(A_3 + A_4\right) \tag{125}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_b}{\tau_b} \left(T_3' - T_2'\right) \underset{3-2}{A} = \left[\rho_a c_a \left(\tau_a - a\right) A_3 + \rho_b c_b \frac{\tau_b}{2} A_4\right] \frac{(T_2' - T_2)}{\Delta t} \tag{126}$$

where

$$A_{1-2} = 0L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_{3-2} = 0L\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)$$

$$A_3 = 0L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_4 = 0L\left(R - \sum \tau_2 - \frac{\tau_b}{4}\right) .$$

Letting $Z = R - \sum \tau_2$ and solving for T_2',

$$T_2' = \frac{\dfrac{T_m k_a \Delta t\left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a\left[\left(Z + \frac{\tau_a - a}{2}\right)\rho_a c_a(\tau_a - a) + \left(Z - \frac{\tau_b}{4}\right)\rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{T_3' k_b \Delta t\left(Z - \frac{\tau_b}{2}\right)\left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b\left[\left(Z + \frac{\tau_a - a}{2}\right)\rho_a c_a(\tau_a - a) + \left(Z - \frac{\tau_b}{4}\right)\rho_b c_b \frac{\tau_b}{2}\right]} + T_2\left(\frac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t\left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a\left[\left(Z + \frac{\tau_a - a}{2}\right)\rho_a c_a(\tau_a - a) + \left(Z - \frac{\tau_b}{4}\right)\rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{k_b \Delta t\left(Z - \frac{\tau_b}{2}\right)\left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_b\left[\left(Z + \frac{\tau_a - a}{2}\right)\rho_a c_a(\tau_a - a) + \left(Z - \frac{\tau_b}{4}\right)\rho_b c_b \frac{\tau_b}{2}\right]} + \dfrac{\tau_a - a}{\tau_a}} \qquad (127)$$

Heat flow in the second thick layer is determined from those equations presented in Section II for a nonrecession case.

e. Special Thick-Thin-Thick with $\delta_a \le \tau_a$ (Figure 26)

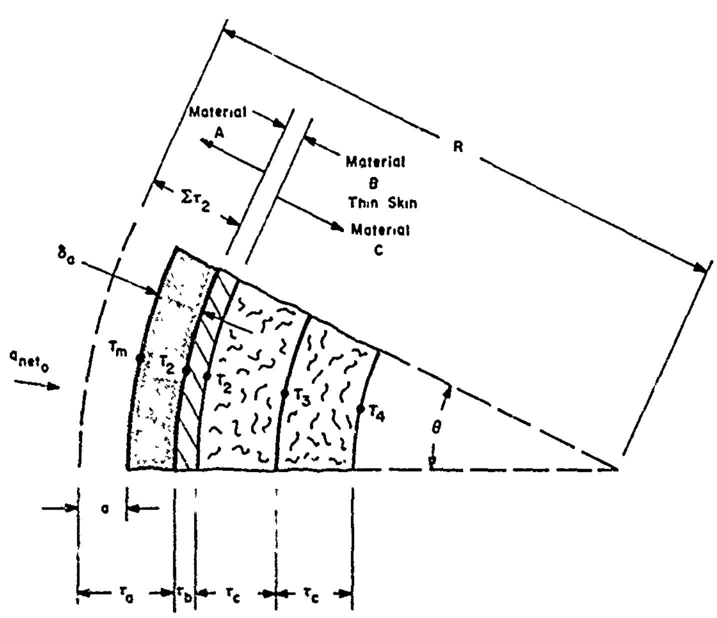


Figure 26.

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$. The energy balance for T_2 is

$$q_{\underset{1\to 2}{cond}} \underset{1-2}{A} + q_{\underset{3\to 2}{cond}} \underset{3-2}{A} = q_{\underset{2}{stored}} \left(A_3 + A_4 + A_5\right) \tag{128}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_c}{\tau_c} \left(T_3' - T_2'\right) \underset{3-2}{A}$$

$$= \left[A_3 \rho_a c_a \frac{\tau_a}{2} + A_4 \rho_b c_b \tau_b + A_5 \rho_c c_c \frac{\tau_c}{2}\right] \frac{(T_2' - T_2)}{\Delta t} \tag{129}$$

where

$$\underset{1-2}{A} = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$\underset{3-2}{A} = \theta L \left(R - \sum \tau_2 - \frac{2\tau_b + \tau_c}{2}\right)$$

$$A_3 = \theta L \left(R - \sum \tau_2 + \frac{\tau_a}{4}\right)$$

$$A_4 = \theta L \left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)$$

$$A_5 = \theta L \left(R - \sum \tau_2 - \tau_b - \frac{\tau_c}{4}\right).$$

Letting

$$Z = R - \sum \tau_2, \quad A = \frac{1}{\theta L}\left[A_3 \rho_a c_a \frac{\tau_a}{2} + A_4 \rho_b c_b \tau_b + A_5 \rho_c c_c \frac{\tau_c}{2}\right],$$

solving for T_2',

$$T_2' = \frac{T_m \dfrac{k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a A} + T_3' \dfrac{k_c \Delta t \left(Z - \frac{2\tau_b + \tau_c}{2}\right) \left(\frac{\tau_a - a}{\tau_a}\right)}{\tau_c A} + T_2 \left(\frac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a A} + \dfrac{k_c \Delta t \left(Z - \frac{2\tau_b + \tau_c}{2}\right)}{\tau_c A} \left(\frac{\tau_a - a}{\tau_a}\right) + \left(\frac{\tau_a - a}{\tau_a}\right)}. \tag{130}$$

(2) $\frac{\tau_a}{2} < a \leq \tau_a$. The energy balance for T_2 is

$$q_{\substack{cond \\ 1 \to 2}} \underset{1-2}{A} + q_{\substack{cond \\ 3 \to 2}} \underset{3-2}{A} = q_{\substack{stored \\ 2}} \left(A_3 + A_4 + A_5\right), \tag{131}$$

or

$$\frac{k_a}{\tau_a - a}\left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_c}{\tau_c}\left(T_3' - T_2'\right) \underset{3-2}{A} = \left[A_3 \rho_a c_a \left(\tau_a - a\right)\right.$$

$$\left. + A_4 \rho_b c_b \tau_b + A_5 \rho_c c_c \frac{\tau_a}{2}\right] \frac{\left(T_2' - T_2\right)}{\Delta t} \tag{132}$$

where

$$\underset{1-2}{A} = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$\underset{3-2}{A} = \theta L \left(R - \sum \tau_2 - \frac{2\tau_b + \tau_c}{2}\right)$$

$$A_3 = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_4 = \theta L \left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)$$

$$A_5 = \theta L \left(R - \sum \tau_2 - \tau_b - \frac{\tau_c}{4}\right) .$$

Letting

$$Z = R - \sum \tau_2 , \quad A = \frac{1}{\theta L}\left[A_3 \rho_a c_a \left(\tau_a - a\right) + A_4 \rho_b c_b \tau_b \right.$$

$$\left. + A_5 \rho_c c_c \frac{\tau_c}{2}\right] ,$$

and solving for T_2',

$$T_2' = \frac{\dfrac{T_m k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a A} + T_3' \dfrac{k_c \Delta t \left(Z - \frac{2\tau_b + \tau_c}{2}\right)}{\tau_c A}\left(\dfrac{\tau_a - a}{\tau_a}\right) + T_2\left(\dfrac{\tau_a - a}{\tau_a}\right)}{\dfrac{k_a \Delta t \left(Z + \frac{\tau_a - a}{2}\right)}{\tau_a A} + \dfrac{k_c \Delta t \left(Z - \frac{2\tau_b + \tau_c}{2}\right)}{\tau_c A}\left(\dfrac{\tau_a - a}{\tau_a}\right) + \dfrac{\tau_a - a}{\tau_a}} . \tag{133}$$

Heat flow in the second thick layer is determined from those equations presented in Section II for a nonrecession case.

3. Sphere Ablation - Conduction

The energy balance for the radial heat flow toward the center in a sphere is basically the same as radial heat flow in a cylinder if the average areas are modified. Appendix A of Report No. RS-TR-65-13 gives some of the derivations for average areas for spheres. Once the ablation front reaches an interior nodal point, the ablation distance "a" is reduced by the value of $\tau_a (0 \le a \le \tau_a)$, and the finite-difference calculation process starts over again. In addition, the radius to the original T_1 nodal point is reduced by the value τ_a.

a. General Thick with $\delta_a > \tau_a$ (Figure 27)

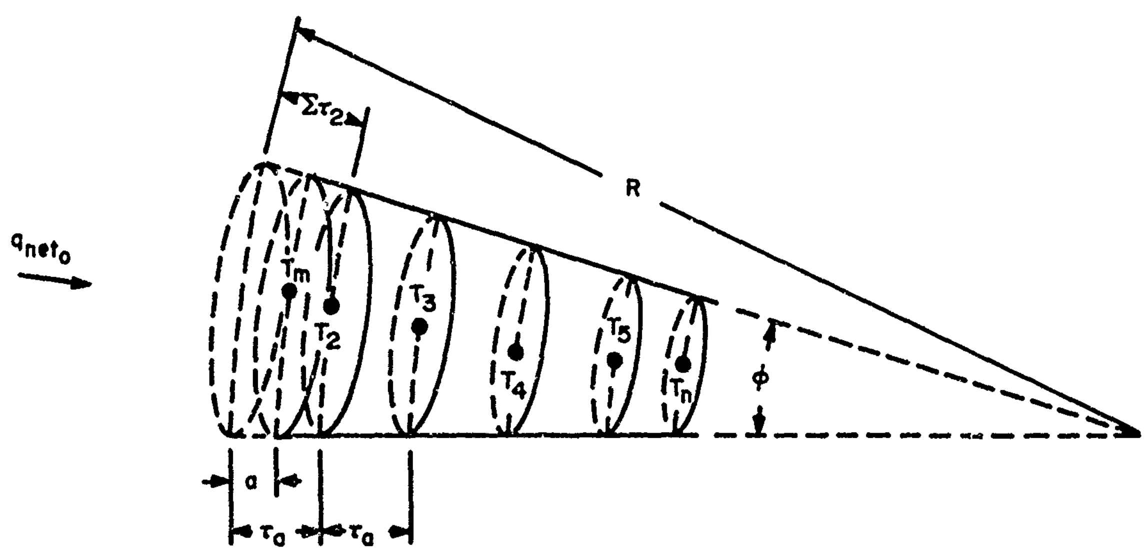


Figure 27.

(1) $0 \le a \le \frac{\tau_a}{2}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} A_{1-2} + q_{\underset{3 \to 2}{cond}} A_{3-2} = q_{\underset{2}{stored}} A_2 \qquad (134)$$

or

$$\frac{k_a}{\tau_a - a}\left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_a}{\tau_a}\left(T_3' - T_2'\right) \underset{3-2}{A}$$

$$= \rho_a \, c_a \, \tau_a \, \frac{\left(T_2' - T_2\right)}{\Delta t} \, A_2 \qquad (135)$$

where

$$\underset{1-2}{A} = \phi\left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$\underset{3-2}{A} = \phi\left[\left(R - \sum \tau_2 - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right]$$

$$A_2 = \phi\left[\left(R - \sum \tau_2\right)^2 + \frac{\tau_a^2}{12}\right] .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad , \qquad Z = R - \sum \tau_2 \ ,$$

and solving for T_2',

$$T_2' = \frac{T_m \beta_a \left[\dfrac{\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right] + T_3' \beta_a \left[\dfrac{\left(Z - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right]\left(\frac{\tau_a - a}{\tau_a}\right) + T_2\left(\frac{\tau_a - a}{\tau_a}\right)}{\beta_a \left[\dfrac{\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right] + \beta_a \left[\dfrac{\left(Z - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right]\left(\frac{\tau_a - a}{\tau_a}\right) + \frac{\tau_a - a}{\tau_a}} \ . \qquad (136)$$

(2) $\underline{\dfrac{\tau_a}{2} < a \le \tau_a .}$ The energy balance for T_2 is

$$\underset{1 \to 2}{q_{cond}} \underset{1-2}{A} + \underset{3 \to 2}{q_{cond}} \underset{3-2}{A} = \underset{2}{q_{stored}} \, A_2 \qquad (137)$$

or

$$\frac{k_a}{(\tau_a - a)}\left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_a}{\tau_a}\left(T_3' - T_2'\right) \underset{3-2}{A}$$

$$= \rho_a \, c_a \left(\frac{3\tau_a}{2} - a\right) \frac{\left(T_2' - T_2\right)}{\Delta t} \, A_2 \qquad (138)$$

where

$$\underset{1-2}{A} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2} \right)^2 + \frac{(\tau_a - a)^2}{12} \right]$$

$$\underset{3-2}{A} = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_a}{2} \right)^2 + \frac{\tau_a^2}{12} \right]$$

$$A_2 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - 2a}{4} \right)^2 + \frac{(3\tau_a - 2a)^2}{48} \right] \quad .$$

Letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad , \qquad Z = R - \sum \tau_2$$

and solving for T_2',

$$T_2' = \frac{\cdot\, T_m \beta_a \left[\dfrac{\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{\left(Z + \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}} \right] + T_3' \beta_a \left[\dfrac{\left(Z - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{\left(Z + \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}} \right] \left(\frac{\tau_a - a}{\tau_a} \right) + T_2 \dfrac{\left(\frac{3\tau_a}{2} - a\right)(\tau_a - a)}{\tau_a^2}}{\beta_a \left[\dfrac{\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{\left(Z + \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}} \right] + \beta_a \left[\dfrac{\left(Z - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{\left(Z + \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}} \right] \left(\frac{\tau_a - a}{\tau_a} \right) + \dfrac{\left(\frac{3\tau_a}{2} - a\right)(\tau_a - a)}{\tau_a^2}} \quad . \tag{139}$$

As long as the remaining wall thickness of the material undergoing recession is equal to one or more than one τ, the interior and interface equations presented in Section II are used to solve the heat balances throughout the structure away from the receding surface.

b. <u>Special Thick with $\delta_a \leq \tau_a$ (Receding Surface $\leq \tau_a$ from Backside) (Figure 28)</u>

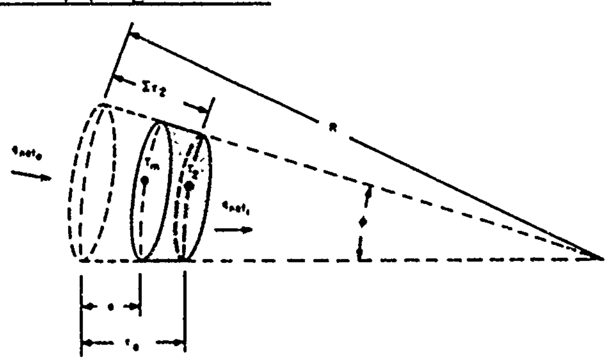


Figure 28.

(1) $\underline{0 \leq a \leq \frac{\tau_a}{2}}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} \underset{1-2}{A} - q_{net_i} A_2 + q_{\underset{2}{stored}} A_3 \tag{140}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} - q_{net_i} A_2$$

$$= \rho_a c_a \frac{\tau_a}{2} A_3 \frac{\left(T_2' - T_2\right)}{\Delta t} \tag{141}$$

where

$$\underset{1-2}{A} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right]$$

$$A_2 = \phi \left[\left(R - \sum \tau_2\right)^2\right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} , \quad Z = R - \sum \tau_2 ,$$

and solving for T_2'

$$T_2' = \frac{T_m (2\beta_a) \left[\dfrac{\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}}{\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}}\right] - \dfrac{2 q_{net_i} Z^2 \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\left[\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \rho_a c_a \tau_a} + T_2 \left(\dfrac{\tau_a - a}{\tau_a}\right)}{2\beta_a \left[\dfrac{\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}}{\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}}\right] + \dfrac{\tau_a - a}{\tau_a}} . \tag{142}$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} \underset{1-2}{A} - q_{net_i} A_2 = q_{\underset{2}{stored}} A_3 \tag{143}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} - q_{net_i} A_2$$

$$= \rho_a c_a \left(\tau_a - a\right) \frac{(T_2' - T_2)}{\Delta t} A_3 \tag{144}$$

where

$$\underset{1-2}{A} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$A_2 = \phi \left[\left(R - \sum \tau_2\right)^2\right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad , \quad Z = R - \sum \tau_2 \quad ,$$

and solving for T_2' ,

$$T_2' = \frac{T_m \beta_a - \dfrac{q_{net_i} \Delta t Z^2 \left(\dfrac{\tau_a - a}{\tau_a}\right)}{\left[\left(Z + \dfrac{\tau_a - a}{2}\right)^2 + \dfrac{(\tau_a - a)^2}{12}\right] \rho_a c_a \tau_a} + T_2 \left(\dfrac{\tau_a - a}{\tau_a}\right)^2}{\beta_a + \left(\dfrac{\tau_a - a}{\tau_a}\right)^2} . \tag{145}$$

c. Special Thick-Thin with $\delta_a \le \tau_a$ (Figure 29)

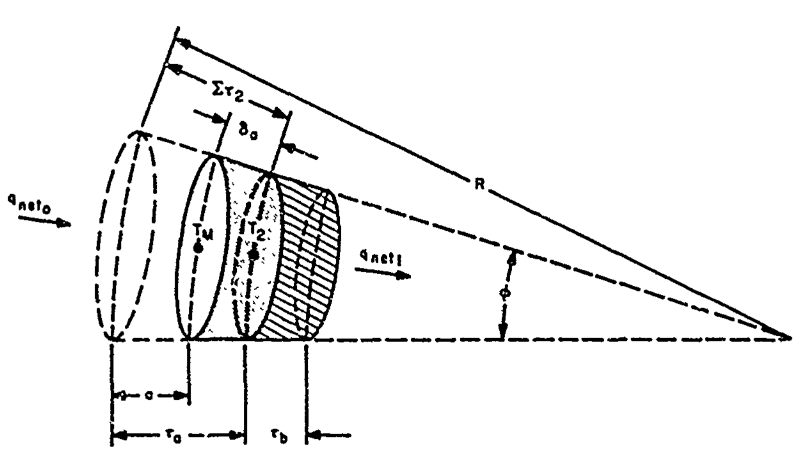


Figure 29.

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$. The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} \, A_{1-2} - q_{net_i} \, A_i = q_{\underset{2}{stored}} \, (A_2 + A_3) \tag{146}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) A_{1-2} - q_{net_i} \, A_i = \left[A_2 \, \rho_a \, c_a \frac{\tau_a}{2} + A_3 \, \rho_b c_b \, \tau_b\right] \frac{(T_2' - T_2)}{\Delta t} \tag{147}$$

where

$$A_{1-2} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$A_i = \phi \left[\left(R - \sum \tau_2 - \tau_b\right)^2\right]$$

$$A_2 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \quad .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a \, c_a \, \tau_a^2} \, , \quad Z = R - \sum \tau_2 \, ,$$

and solving for T_2',

$$T_2' = \frac{\dfrac{T_m \dfrac{k_a \Delta t}{\tau_a} \left[\left(Z + \dfrac{\tau_a - a}{2}\right)^2 + \dfrac{(\tau_a - a)^2}{12}\right] - q_{net_i} \left(Z - \tau_b\right)^2 \Delta t \left(\dfrac{\tau_a - a}{\tau_a}\right)}{\left\{\left[\left(Z + \dfrac{\tau_a}{4}\right)^2 + \dfrac{\tau_a^2}{48}\right] \rho_a \, c_a \dfrac{\tau_a}{2} + \left[\left(Z - \dfrac{\tau_b}{2}\right)^2 + \dfrac{\tau_b^2}{12}\right] \rho_b \, c_b \, \tau_b\right\}}}{\dfrac{\tau_a - a}{\tau_a} + \dfrac{\dfrac{k_a \Delta t}{\tau_a} \left[\left(Z + \dfrac{\tau_a - a}{2}\right)^2 + \dfrac{(\tau_a - a)^2}{12}\right]}{\left\{\left[\left(Z + \dfrac{\tau_a}{4}\right)^2 - \dfrac{\tau_a^2}{48}\right] \rho_a \, c_a \dfrac{\tau_a}{2} + \left[\left(Z - \dfrac{\tau_b}{2}\right)^2 + \dfrac{\tau_b^2}{12}\right] \rho_b \, c_b \, \tau_b\right\}}} \tag{148}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance for T_2' is

$$q_{\underset{1 \to 2}{cond}} \, A_{1-2} - q_{net_i} \, A_i = q_{\underset{2}{stored}} \, (A_2 + A_3) \tag{149}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} - q_{net_i} \; A_i = \left[A_2 \, \rho_a \, c_a \left(\tau_a - a\right) + A_3 \rho_b \; c_b \, \tau_b\right] \frac{(T_2' - T_2)}{\Delta t} \tag{150}$$

where

$$\underset{1-2}{A} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$A_i = \phi \left[\left(R - \sum \tau_2 - \tau_b\right)^2\right]$$

$$A_2 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] .$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad , \quad Z = R - \sum \tau_2 \; ,$$

and solving for T_2',

$$T_2' = \frac{T_2\left(\frac{\tau_a - a}{\tau_a}\right) + \dfrac{T_m \frac{k_a \Delta t}{\tau_a} \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] - q_{net_1} \left(Z - \tau_b\right)^2 \Delta t \left(\frac{\tau_a - a}{\tau_a}\right)}{\left\{\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \rho_a \, c_a \left(\tau_a - a\right) + \left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \rho_b \, c_b \, \tau_b\right\}}}{\frac{\tau_a - a}{\tau_a} + \frac{k_a \Delta t}{\tau_a} \dfrac{\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]}{\left\{\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \rho_a \, c_a \left(\tau_a - a\right) + \left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \rho_b \, c_b \, \tau_b\right\}}} . \tag{151}$$

d. <u>Special Thick-Thick with $\delta_a \leq \tau_a$ (Figure 30)</u>

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_2 is

$$\underset{1 \to 2}{q_{cond}} \; \underset{1-2}{A} + \underset{3 \to 2}{q_{cond}} \; \underset{3-2}{A} = \underset{2}{q_{stored}} \left(A_3 + A_4\right) \tag{152}$$

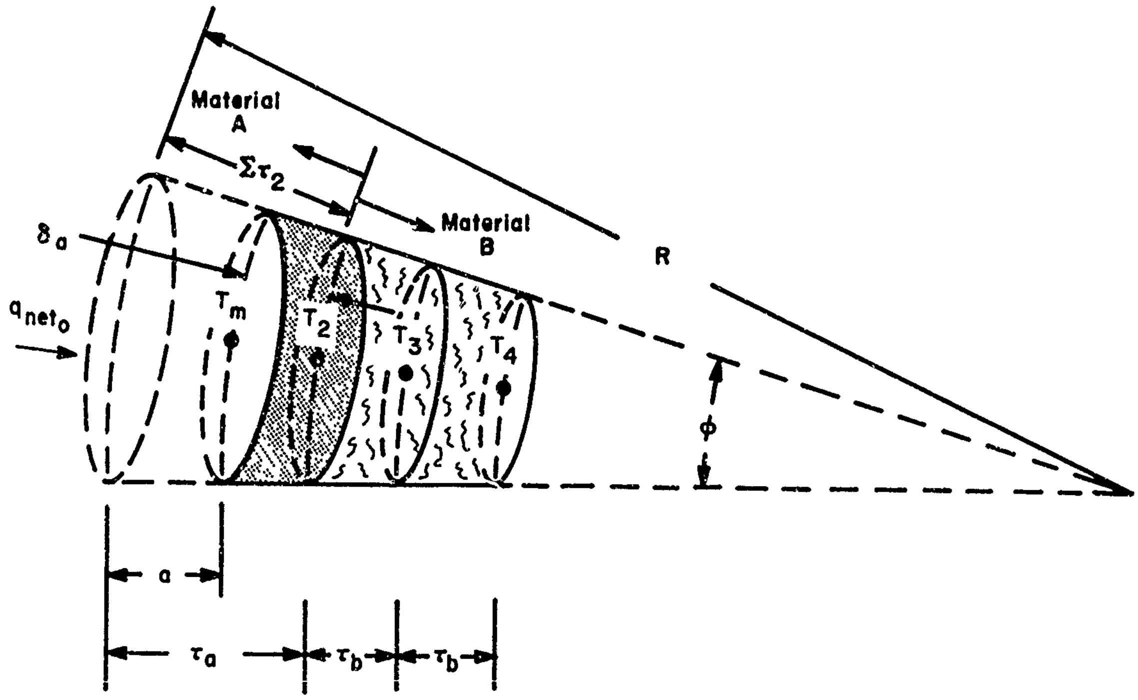


Figure 30.

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_b}{\tau_b} \left(T_3' - T_2'\right) \underset{3-2}{A} = \left[\rho_a c_a \frac{\tau_a}{2} A_3 + \rho_b c_b \frac{\tau_b}{2} A_4\right] \frac{(T_2' - T_2)}{\Delta t} \tag{153}$$

where

$$\underset{1-2}{A} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$\underset{3-2}{A} = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]$$

$$A_4 = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right] .$$

Letting $Z = R - \sum \tau_2$ and solving for T_2',

$$T_2' = \frac{T_2\left(\frac{\tau_a - a}{\tau_a}\right) + \dfrac{T_m \frac{k_a \Delta t}{\tau_a}\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] + T_3' \frac{k_b \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)\left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\left\{\left[\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]\rho_a c_a \frac{\tau_a}{2} + \left[\left(Z - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\rho_b c_b \frac{\tau_b}{2}\right\}}}{\frac{\tau_a - a}{\tau_a} + \dfrac{\frac{k_a \Delta t}{\tau_a}\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] + \frac{k_b \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)\left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\left\{\left[\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right]\rho_a c_a \frac{\tau_a}{2} + \left[\left(Z - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\rho_b c_b \frac{\tau_b}{2}\right\}}} \quad (154)$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a.}$ The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} \, A_{1-2} + q_{\underset{3 \to 2}{cond}} \, A_{3-2} = q_{\underset{2}{stored}} \, (A_3 + A_4) \quad (155)$$

or

$$\frac{k_a}{(\tau_a - a)}\left(T_m - T_2'\right) A_{1-2} + \frac{k_b}{\tau_b}\left(T_3' - T_2'\right) A_{3-2} = \left[\rho_a c_a \left(\tau_a - a\right) A_3 + \rho_b c_b \frac{\tau_b}{2} A_4\right] \frac{(T_2' - T_2)}{\Delta t} \quad (156)$$

where

$$A_{1-2} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$A_{3-2} = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$A_4 = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right] .$$

Letting $Z = R - \sum \tau_2$, and solving for T_2',

$$T_2' = \frac{T_2\left(\frac{\tau_a - a}{\tau_a}\right) + \dfrac{T_m \frac{k_a \Delta t}{\tau_a}\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] + T_3' \frac{k_b \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)\left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\left\{\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]\rho_a c_a \left(\tau_a - a\right) + \left[\left(Z - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\rho_b c_b \frac{\tau_b}{2}\right\}}}{\frac{\tau_a - a}{\tau_a} + \dfrac{\frac{k_a \Delta t}{\tau_a}\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] + \frac{k_b \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)\left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]}{\left\{\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]\rho_a c_a \left(\tau_a - a\right) + \left[\left(Z - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right]\rho_b c_b \frac{\tau_b}{2}\right\}}} \quad (157)$$

Heat flow in the second thick layer is determined from those equations presented in Section II for a nonrecession case.

e. Special Thick-Thin-Thick with $\delta_a \leq \tau_a$ (Figure 31)

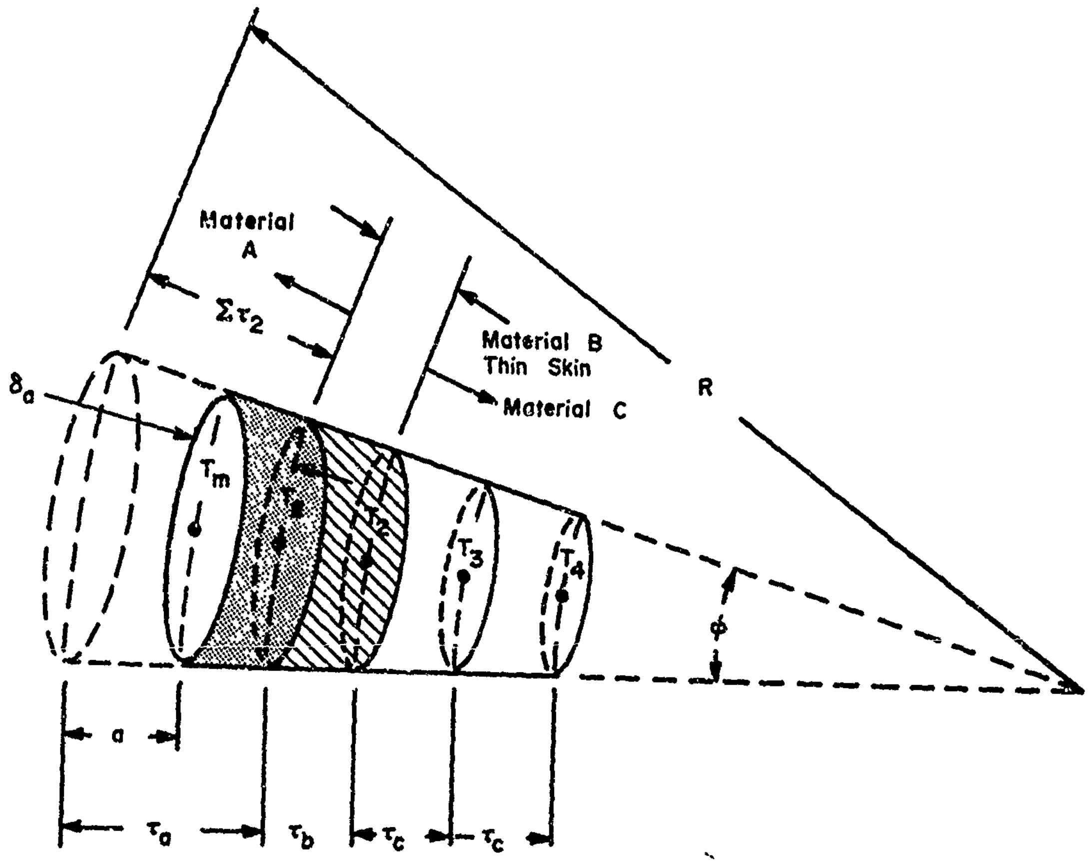


Figure 31.

(1) $0 \leq a \leq \dfrac{\tau_a}{2}$. The energy balance for T_2 is

$$q_{\underset{1\rightarrow 2}{cond}} \; A_{1-2} + q_{\underset{3\rightarrow 2}{cond}} \; A_{3-2} = q_{\underset{2}{stored}} \, (A_3 + A_4 + A_5) \tag{158}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) A_{1-2} + \frac{k_c}{\tau_c} \left(T_3' - T_2'\right) A_{3-2} = \left[A_3 \, \rho_a \, c_a \frac{\tau_a}{2} + A_4 \, \rho_b \, c_b \, \tau_b + A_5 \, \rho_c \, c_c \frac{\tau_c}{2}\right] \frac{(T_2' - T_2)}{\Delta t} \tag{159}$$

where

$$A_{1-2} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2} \right)^2 + \frac{(\tau_a - a)^2}{12} \right]$$

$$A_{3-2} = \phi \left[\left(R - \sum \tau_2 - \tau_b - \frac{\tau_c}{2} \right)^2 + \frac{\tau_c^2}{12} \right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a}{4} \right)^2 + \frac{\tau_a^2}{48} \right]$$

$$A_4 = \phi \left[\left(R - \sum \tau_2 - \frac{\tau_b}{2} \right)^2 + \frac{\tau_b^2}{12} \right]$$

$$A_5 = \phi \left[\left(R - \sum \tau_2 - \tau_b - \frac{\tau_c}{4} \right)^2 + \frac{\tau_c^2}{48} \right] .$$

Letting $Z = R - \sum \tau_2$ and solving for T_2',

$$T_2' = \frac{T_2 \left(\frac{\tau_a - a}{\tau_a} \right) + \dfrac{T_m \frac{k_a \Delta t}{\tau_a} \left[\left(Z + \frac{\tau_a - a}{2} \right)^2 + \frac{(\tau_a - a)^2}{12} \right] + T_3' \frac{k_c \Delta t}{\tau_c} \left(\frac{\tau_a - a}{\tau_a} \right) \left[\left(Z - \tau_b - \frac{\tau_c}{2} \right)^2 + \frac{\tau_c^2}{12} \right]}{\left\{ \left[\left(Z + \frac{\tau_a}{4} \right)^2 + \frac{\tau_a^2}{48} \right] \rho_a c_a \frac{\tau_a}{2} + \left[\left(Z - \frac{\tau_b}{2} \right)^2 + \frac{\tau_b^2}{12} \right] \rho_b c_b \tau_b + \left[\left(Z - \tau_b - \frac{\tau_c}{4} \right)^2 + \frac{\tau_c^2}{48} \right] \rho_c c_c \frac{\tau_c}{2} \right\}}}{\frac{\tau_a - a}{\tau_a} + \dfrac{\frac{k_a \Delta t}{\tau_a} \left[\left(Z + \frac{\tau_a - a}{2} \right)^2 + \frac{(\tau_a - a)^2}{12} \right] + \frac{k_c \Delta t}{\tau_c} \left(\frac{\tau_a - a}{\tau_a} \right) \left[\left(Z - \tau_b - \frac{\tau_c}{2} \right)^2 + \frac{\tau_c^2}{12} \right]}{\left\{ \left[\left(Z + \frac{\tau_a}{4} \right)^2 + \frac{\tau_a^2}{48} \right] \rho_a c_a \frac{\tau_a}{2} + \left[\left(Z - \frac{\tau_b}{2} \right)^2 + \frac{\tau_b^2}{12} \right] \rho_b c_b \tau_b + \left[\left(Z - \tau_b - \frac{\tau_c}{4} \right)^2 + \frac{\tau_c^2}{48} \right] \rho_c c_c \frac{\tau_c}{2} \right\}}} . \quad (160)$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$. The energy balance for T_2' is

$$q_{\underset{1 \to 2}{cond}} A_{1-2} + q_{\underset{3 \to 2}{cond}} A_{3-2} = q_{\underset{2}{stored}} (A_3 + A_4 + A_5) \quad (161)$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2' \right) A_{1-2} + \frac{k_c}{\tau_c} \left(T_3' - T_2' \right) A_{3-2} = \left[A_3 \rho_a c_a \left(\tau_a - a \right) \right.$$

$$\left. + A_4 \rho_b c_b \tau_b + A_5 \rho_c c_c \frac{\tau_c}{2} \right] \frac{(T_2' - T_2)}{\Delta t} \quad (162)$$

where

$$A_{1-2} = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2} \right)^2 + \frac{(\tau_a - a)^2}{12} \right]$$

$$A_{3-2} = \phi \left[\left(R - \sum \tau_2 - \tau_b - \frac{\tau_c}{2} \right)^2 + \frac{\tau_c^2}{12} \right]$$

$$A_3 = \phi\left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right]$$

$$A_4 = \phi\left[\left(R - \sum \tau_2 - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right]$$

$$A_5 = \phi\left[\left(R - \sum \tau_2 - \tau_b - \frac{\tau_c}{4}\right)^2 + \frac{\tau_c^2}{48}\right] .$$

Letting $Z = R - \sum \tau_2$ and

$$A = \frac{A_3}{\phi}\, \rho_a\, c_a \left(\tau_a - a\right) + \frac{A_4}{\phi}\, \rho_b\, c_b\, \tau_b + \frac{A_5}{\phi}\, \rho_c\, c_c\, \frac{\tau_c}{2} ,$$

and solving for T_2'

$$T_2' = \frac{T_2\left(\frac{\tau_a - a}{\tau_a}\right) + \dfrac{T_m \frac{k_a \Delta t}{\tau_a}\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] + T_3' \frac{k_c \Delta t}{\tau_c}\left(\frac{\tau_a - a}{\tau_a}\right)\left[\left(Z - \tau_b - \frac{\tau_c}{2}\right)^2 + \frac{\tau_c^2}{12}\right]}{A}}{\frac{\tau_a - a}{\tau_a} + \dfrac{\frac{k_a \Delta t}{\tau_a}\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] + \frac{k_c \Delta t}{\tau_c}\left(\frac{\tau_a - a}{\tau_a}\right)\left[\left(Z - \tau_b - \frac{\tau_c}{2}\right)^2 + \frac{\tau_c^2}{12}\right]}{A}} . \tag{163}$$

Heat flow in the second thick layer is determined from those equations presented in Section II for a nonrecession case.

Section IV. HEAT CONDUCTION AFTER ABLATION TERMINATES (T_1' and T_2')

With the equations derived for heat conduction prior to ablation and during ablation, the heat conduction equations after ablation stops must be considered. The equations for conduction prior to ablation cannot be used after ablation ceases without some modification to the temperature grid because there is no assurance that the ablation into a new τ will be zero, i.e., the ablation usually will not cease with the receded surface coinciding with an original temperature node location. Although it is possible to select a new temperature grid system for the material left after surface recession ceases and to obtain proper temperature of new nodal points based on interpolations from the calculated temperature gradient when ablation ceases, an approach is taken herein whereby the original grid remains unchanged. Heat conduction equations for T_3' through T_n' are derived in the same manner as before and during ablation. The equations used to calculate T_1' and T_2' are derived in Paragraphs 1, 2, and 3.

1. Flat Plate

a. General Thick (Figure 16, T_m Taking on the Value T_1') with $\delta_a > \tau_a$

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is

$$q_{net_o} A - q_{\underset{1\to 2}{cond}} A = q_{\underset{1}{stored}} A \; . \tag{164}$$

For the flat plate conduction, A may be taken as unity; then,

$$q_{net_o} - \frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) = \rho_a \, c_a \left(\frac{\tau_a}{2} - a\right) \frac{T_1' - T_1}{\Delta t} \; . \tag{165}$$

The energy balance for T_2' may be taken as

$$q_{\underset{1\to 2}{cond}} A + q_{\underset{3\to 2}{cond}} A = q_{\underset{2}{stored}} A \tag{166}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) = \rho_a \, c_a \, \tau_a \, \frac{(T_2' - T_2)}{\Delta t} \; . \tag{167}$$

This equation is the same as Equation (74) while the surface is ablating, except for T_m taking on the value T_1'.

Since T_3' through T_n', including any interfaces, can be calculated by using ordinary forward finite-difference methods, Equations (165) and (167) have two unknowns. Solving Equations (165) and (167) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

results in

$$T_1' = \frac{\left\{\frac{\tau_a - a}{\tau_a} + \beta_a \left[1 + \frac{\tau_a - a}{\tau_a}\right]\right\} \left\{T_1 \frac{\frac{\tau_a}{2} - a}{\tau_a} + \frac{q_{net_o} \, \Delta t}{\rho_a \, c_a \, \tau_a}\right\} + \beta_a \left[T_2 + T_3' \beta_a\right]}{\left[\frac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\right] \left[1 + \beta_a\right] + \beta_a \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} \tag{168}$$

and

$$T_2' = \frac{\left[\frac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\right] \left[T_2 + \beta_a T_3'\right] + \beta_a \frac{q_{net_o} \, \Delta t}{\rho_a \, c_a \, \tau_a} + T_1 \beta_a \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)}{\left[\frac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\right] \left[1 + \beta_a\right] + \beta_a \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} \; . \tag{169}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance for T_1' is

$$q_{net_o} \, A - \underset{1 \to 2}{q_{cond}} \, A = \underset{1}{q_{stored}} \, A. \tag{170}$$

For the flat-plate conduction, A may be taken as unity; then,

$$q_{net_o} - \frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) = 0 \; . \tag{171}$$

The energy balance for T_2' may be taken as

$$\underset{1 \to 2}{q_{cond}} \, A + \underset{3 \to 2}{q_{cond}} \, A = \underset{2}{q_{stored}} \, A \tag{172}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right)$$

$$= \rho_a \, c_a \left(\frac{3\tau_a}{2} - a\right) \frac{(T_2' - T_2)}{\Delta t} \; . \tag{173}$$

This equation is the same as Equation (78) while the surface is ablating, except for T_m taking on the value T_1'. Solving Equations (171) and (173) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

gives

$$T_1' = \frac{q_{net_o}\left(\frac{\tau_a}{k_a}\right)\left[\frac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a} + \beta_a + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right)\right] + T_2\left(\frac{\frac{3\tau_a}{2} - a}{\tau_a}\right) + T_3'\beta_a}{\frac{\frac{3\tau_a}{2} - a}{\tau_a} + \beta_a} \tag{174}$$

and

$$T_2' = \frac{q_{net_o}\left(\frac{\tau_a}{k_a}\right)\beta_a + T_2\left(\frac{\frac{3\tau_a}{2} - a}{\tau_a}\right) + T_3'\beta_a}{\frac{\frac{3\tau_a}{2} - a}{\tau_a} + \beta_a} \; . \tag{175}$$

It should be noted that the denominators of Equations (168) and (169) are identical as are the denominators of Equations (174) and (175).

When "a" $\rightarrow \tau_a$, Equations (174) and (175) for T_1' and T_2' reduce as expected to

$$T_1' = T_2' = q_{net_o}\frac{\tau_a}{k_a} + T_3' \; . \tag{176}$$

b. <u>Special Thick (Receding Surface $\leq \tau_a$ from Backside, Figure 18, T_m Taking on Value T_1') with $\delta_a \leq \tau_a$</u>

When the receding surface is less than one incremental τ from another material surface or backside, special considerations must be made for all material combinations normally experienced.

(1) <u>$0 \leq a \leq \frac{\tau_a}{2}$</u>. The energy balance for T_1' is

$$q_{net_o} A - q_{\underset{1 \rightarrow 2}{cond}} A^{\cdot} = q_{\underset{1}{stored}} A \; . \tag{177}$$

For the flat-plate conduction, A may be taken as unity; then,

$$q_{net_o} - \frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) = \rho_a \, c_a \left(\frac{\tau_a}{2} - a\right) \frac{\left(T_1' - T_1\right)}{\Delta t} . \tag{178}$$

The energy balance for T_2' may be taken as

$$\underset{1 \to 2}{q_{cond}} \, A - q_{net_i} \, A = \underset{2}{q_{stored}} \, A \tag{179}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) - q_{net_i} = \rho_a \, c_a \, \frac{\tau_a}{2} \; \frac{\left(T_2' - T_2\right)}{\Delta t} . \tag{180}$$

Solving Equations (178) and (180) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a{}^2}$$

gives

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} + 2\beta_a\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o} \, \Delta t}{\rho_a \, c_a \, \tau_a}\right] + \beta_a \left[T_2 - \frac{2 \, q_{net_i} \, \Delta t}{\rho_a \, c_a \, \tau_a}\right]}{\frac{\left(\frac{\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a{}^2} + \beta_a + 2\beta_a \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} \tag{181}$$

and

$$T_2' = \frac{2\beta_a \left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o} \, \Delta t}{\rho_a \, c_a \, \tau_a}\right] + \left[\frac{\left(\frac{\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a{}^2} + \beta_a\right]\left[T_2 - \frac{2 \, q_{net_i} \, \Delta t}{\rho_a \, c_a \, \tau_a}\right]}{\frac{\left(\frac{\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a{}^2} + \beta_a + 2\beta_a \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} . \tag{182}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a.}$ The energy balance for T_1' is

$$q_{net_o} \, A - \underset{1 \to 2}{q_{cond}} \, A = \underset{1}{q_{stored}} \, A \tag{183}$$

For the flat-plate conduction, A may be taken as unity; then,

$$q_{net_o} - \frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) \quad 0 . \tag{184}$$

The energy balance for T_2' is

$$q_{\substack{cond \\ 1 \to 2}} A - q_{net_i} A = q_{\substack{stored \\ 2}} A \tag{185}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) - q_{net_i} = \rho_a c_a \left(\frac{\tau_a - a}{2}\right) \frac{\left(T_2' - T_2\right)}{\Delta t} . \tag{186}$$

Solving Equations (184) and (186) simultaneously and letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2}$$

results in

$$T_1' = T_2 + \frac{\tau_a}{\left(\tau_a - a\right)} \left[q_{net_o} \left(\frac{\tau_a}{k_a}\right) \left\{ \beta_a + \left(\frac{\tau_a - a}{\tau_a}\right)^2 \right\} - \frac{q_{net_i} \Delta t}{\rho_a c_a \tau_a} \right] \tag{187}$$

and

$$T_2' = T_2 + \frac{\tau_a}{\left(\tau_a - a\right)} \left[q_{net_o} \left(\frac{\tau_a}{k_a}\right) \beta_a - \frac{q_{net_i} \Delta t}{\rho_a c_a \tau_a} \right] . \tag{188}$$

c. Special Thick-Thin (Figure 19, T_m Taking on Value T_1') with $\delta_a \le \tau_a$

(1) $0 \le a \le \frac{\tau_a}{2}$. The energy balance for T_1' is

$$q_{net_o} A - q_{\substack{cond \\ 1 \to 2}} A = q_{\substack{stored \\ 1}} A . \tag{189}$$

For the flat-plate conduction, A may be taken as unity; then,

$$q_{net_o} - \frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) = \rho_a c_a \left(\frac{\tau_a}{2} - a\right) \frac{\left(T_1' - T_1\right)}{\Delta t} . \tag{190}$$

The energy balance for T_2' may be taken as

$$q_{\underset{1\rightarrow 2}{cond}} A - q_{net_i} A = q_{\underset{2}{stored}} A \tag{191}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) - q_{net_i} = \left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right) \frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{192}$$

Solving Equations (190) and (192) simultaneously and letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2}$$

gives

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} + \frac{k_a \Delta t}{\tau_a\left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right)}\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o} \Delta t}{\rho_a c_a \tau_a}\right] + \beta_a \left[T_2 - \frac{q_{net_i} \Delta t}{\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b}\right]}{\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a + \frac{k_a \Delta t\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)}{\tau_a\left[\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right]}} \tag{193}$$

and

$$T_2' = \frac{\left[\frac{k_a \Delta t}{\tau_a\left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right)}\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o} \Delta t}{\rho_a c_a \tau_a}\right] + \left[\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\right]\left[T_2 - \frac{q_{net_i} \Delta t}{\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b}\right]}{\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a + \frac{k_a \Delta t\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)}{\tau_a\left(\rho_a c_a \frac{\tau_a}{2} + \rho_b c_b \tau_b\right)}} \quad . \tag{194}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance for T_1' is

$$q_{net_o} A - q_{\underset{1\rightarrow 2}{cond}} A = q_{\underset{1}{stored}} A \quad . \tag{195}$$

For the flat-plate conduction, A may be taken as unity; then,

$$q_{net_o} - \frac{k_a}{(\tau_a - a)} \left(T_1' - T_2'\right) = 0 \quad . \tag{196}$$

The energy balance for T_2' is

$$q_{\substack{cond\\1\to 2}} A - q_{net_i} A = q_{\substack{stored\\2}} A \tag{197}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) - q_{net_i} = \left[\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \tau_b\right] \frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{198}$$

Solving Equations (196) and (198) simultaneously gives

$$T_1' = T_2 + q_{net_o}\left[\frac{\Delta t}{\left\{\rho_a c_a (\tau_a - a) + \rho_b c_b \tau_b\right\}} + \frac{\tau_a - a}{k_a}\right] - \frac{q_{net_i} \Delta t}{\left\{\rho_a c_a (\tau_a - a) + \rho_b c_b \tau_b\right\}} \tag{199}$$

and

$$T_2' = T_2 + \frac{\Delta t \left[q_{net_o} - q_{net_i}\right]}{\left\{\rho_a c_a (\tau_a - a) + \rho_b c_b \tau_b\right\}} \quad . \tag{200}$$

d. <u>Special Thick-Thick (Figure 20, T_m Taking on Value T_1') with $\delta_a \leq \tau_a$</u>

(1) <u>$0 \leq a \leq \frac{\tau_a}{2}$.</u> The energy balance for T_1' is

$$q_{net_o} A - q_{\substack{cond\\1\to 2}} A = q_{\substack{stored\\1}} A \tag{201}$$

or

$$q_{net_o} - \frac{k_a \left(T_1' - T_2'\right)}{\left(\tau_a - a\right)} = \rho_a c_a \left(\frac{\tau_a}{2} - a\right) \frac{\left(T_1' - T_1\right)}{\Delta t} \quad . \tag{202}$$

The energy balance for T_2' may be taken as

$$q_{\underset{1 \to 2}{cond}} A + q_{\underset{3 \to 2}{cond}} A = q_{\underset{2}{stored}} A \tag{203}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) + \frac{k_b}{\tau_b} \left(T_3' - T_2'\right)$$

$$= \left(\rho_a\, c_a \frac{\tau_a}{2} + \rho_b\, c_b \frac{\tau_b}{2}\right) \frac{\left(T_2' - T_2\right)}{\Delta t} \tag{204}$$

T_3' through T_n' can be calculated using the standard forward finite-difference equations given in Section III on heat conduction prior to ablation.

Solving Equations (202) and (204) simultaneously and letting

$$\beta_a = \frac{k_a\, \Delta t}{\rho_a\, c_a\, \tau_a^2}$$

results in

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} + \frac{\frac{2k_a\, \Delta t}{\tau_a} + \frac{2k_b\, \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)}{\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o}\, \Delta t}{\rho_a\, c_a\, \tau_a}\right] + \beta_a\left[T_2 + T_3'\, \frac{2k_b\, \Delta t}{\tau_b\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}\right]}{\beta_a\left[1 + \frac{2k_b\, \Delta t}{\tau_b\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}\right] + \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left[\frac{\tau_a - a}{\tau_a} + \frac{\frac{2k_a\, \Delta t}{\tau_a} + \frac{2k_b\, \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)}{\left(\rho_a c_a\, \tau_a + \rho_b c_b\, \tau_b\right)}\right]} \tag{205}$$

and

$$T_2' = \frac{\frac{2k_a\, \Delta t}{\tau_a\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{a}\right) + \frac{q_{net_o}\, \Delta t}{\rho_a\, c_a\, \tau_a}\right] + \left[\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\right]\left[T_2 + T_3'\, \frac{2k_b\, \Delta t}{\tau_b\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}\right]}{\beta_a\left[1 + \frac{2k_b\, \Delta t}{\tau_b\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}\right] + \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left[\frac{\tau_a - a}{\tau_a} + \frac{\frac{2k_a\, \Delta t}{\tau_a} + \frac{2k_b\, \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)}{\left(\rho_a\, c_a\, \tau_a + \rho_b\, c_b\, \tau_b\right)}\right]} . \tag{206}$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a.}$ The energy balance for T_1' is

$$q_{net_o} A - q_{\underset{1 \to 2}{cond}} A = q_{\underset{1}{stored}} A \ . \tag{207}$$

Then

$$\dot{q}_{net_o} - \frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) = 0 \quad . \tag{208}$$

The energy balance for T_2' is

$$q_{\underset{1 \to 2}{cond}} A + q_{\underset{3 \to 2}{cond}} A = q_{\underset{2}{stored}} A \tag{209}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) + \frac{k_b}{\tau_b} \left(T_3' - T_2'\right) = \left[\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \frac{\tau_b}{2}\right] \frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{210}$$

Solving Equations (208) and (210) simultaneously gives

$$T_1' = \frac{q_{net_o}\left(\frac{\tau_a}{k_a}\right)\left[\frac{\tau_a - a}{\tau_a} + \frac{\frac{k_a \Delta t}{\tau_a} + \frac{k_b \Delta t}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)}{\left[\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \frac{\tau_b}{2}\right]}\right] + T_2 + T_3' \frac{k_b \Delta t}{\tau_b \left[\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \frac{\tau_b}{2}\right]}}{1 + \frac{k_b \Delta t}{\tau_b \left\{\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \frac{\tau_b}{2}\right\}}} \tag{211}$$

and

$$T_2' = \frac{\frac{q_{net_o} \Delta t}{\left[\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \frac{\tau_b}{2}\right]} + T_2 + T_3' \frac{k_b \Delta t}{\tau_b \left\{\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \frac{\tau_b}{2}\right\}}}{1 + \frac{k_b \Delta t}{\tau_b \left\{\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \frac{\tau_b}{2}\right\}}} \quad . \tag{212}$$

e. <u>Special Thick-Thin-Thick (Figure 21, T_m Taking on Value T_1') with $\delta_a < \tau_a$</u>

T_3' through T_n' is first calculated using the forward finite-difference equation.

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is

$$q_{net_o} A - q_{\underset{1 \to 2}{cond}} A = q_{\underset{1}{stored}} A \tag{213}$$

or

$$q_{net_o} - \frac{k_a}{\left(\tau_a - a\right)}\left(T_1' - T_2'\right) = \rho_a\, c_a\left(\frac{\tau_a}{2} - a\right)\frac{\left(T_1' - T_1\right)}{\Delta t} \quad . \tag{214}$$

The energy balance for T_2' is

$$q_{\underset{1\rightarrow 2}{cond}}\, A + q_{\underset{3\rightarrow 2}{cond}}\, A = q_{\underset{2}{stored}}\, A \tag{215}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)}\left(T_1' - T_2'\right) + \frac{k_c}{\tau_c}\left(T_3' - T_2'\right) = \left[\rho_a\, c_a\,\frac{\tau_a}{2} + \rho_b\, c_b\, \tau_b + \rho_c\, c_c\,\frac{\tau_c}{2}\right]\frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{216}$$

Solving Equations (214) and (216) simultaneously and letting

$$B = \rho_a\, c_a\,\frac{\tau_a}{2} + \rho_b\, c_b\, \tau_b + \rho_c\, c_c\,\frac{\tau_c}{2}$$

and

$$\beta_a = \frac{k_a\, \Delta t}{\rho_a\, c_a\, \tau_a^{\,2}}$$

gives

$$T_1' = \frac{\left[\dfrac{\tau_a - a}{\tau_a} + \dfrac{\dfrac{k_a\,\Delta t}{\tau_a} + \dfrac{k_a\,\Delta t}{\tau_c}\left(\dfrac{\tau_a - a}{\tau_a}\right)}{B}\right]\left[T_1\left(\dfrac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \dfrac{q_{net_o}\,\Delta t}{\rho_a\, c_a\, \tau_a}\right] + \beta_a\left[T_2 + T_3'\,\dfrac{k_c\,\Delta t}{\tau_c\, B}\right]}{\beta_a\left[1 + \dfrac{k_c\,\Delta t}{\tau_c\, B}\right] + \left(\dfrac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left[\dfrac{\tau_a - a}{\tau_a} + \dfrac{\dfrac{k_a\,\Delta t}{\tau_a} + \dfrac{k_c\,\Delta t}{\tau_c}\left(\dfrac{\tau_a - a}{\tau_a}\right)}{B}\right]} \tag{217}$$

and

$$T_2' = \frac{\dfrac{k_a\,\Delta t}{\tau_a\, B}\left[T_1\left(\dfrac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \dfrac{q_{net_o}\,\Delta t}{\rho_a\, c_a\, \tau_a}\right] + \left[\dfrac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^{\,2}} + \beta_a\right]\left[T_2 + T_3'\,\dfrac{k_c\,\Delta t}{\tau_c\, B}\right]}{\beta_a\left[1 + \dfrac{k_c\,\Delta t}{\tau_c\, B}\right] + \left(\dfrac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left[\dfrac{\tau_a - a}{\tau_a} + \dfrac{\dfrac{k_a\,\Delta t}{\tau_a} + \dfrac{k_c\,\Delta t}{\tau_c}\left(\dfrac{\tau_a - a}{\tau_a}\right)}{B}\right]} \tag{218}$$

(2) $\frac{\tau_a}{2} < a \le \tau_a$. The energy balance for T_1' is

$$q_{net_o} A - q_{\underset{1 \to 2}{cond}} A = q_{\underset{1}{stored}} A \tag{219}$$

or

$$q_{net_o} - \frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) = 0 \quad . \tag{220}$$

The energy balance for T_2' is

$$q_{\underset{1 \to 2}{cond}} A + q_{\underset{3 \to 2}{cond}} A = q_{\underset{2}{stored}} A \tag{221}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) + \frac{k_c}{\tau_c} \left(T_3' - T_2'\right) = \left[\rho_a c_a \left(\tau_a - a\right) + \rho_b c_b \tau_b + \rho_c c_c \frac{\tau_c}{2}\right] \frac{\left(T_2' - T_2\right)}{\Delta t} \quad . \tag{222}$$

Solving Equations (220) and (222) simultaneously and letting

$$B = \rho a \, c a \left(\tau_a - a\right) + \rho b \, c b \, \tau_b + \rho_c \, c_c \frac{\tau_c}{2}$$

results in

$$T_1' = \frac{q_{net_o}\left(\frac{\tau_a}{k_a}\right) \left[\frac{\tau_a - a}{\tau_a} + \frac{\frac{k_a \Delta t}{\tau_a} + \frac{k_c \Delta t}{\tau_c} \left(\frac{\tau_a - a}{\tau_a}\right)}{B}\right] + T_2 + T_3' \frac{k_c \Delta t}{\tau_c B}}{1 + \frac{k_c \Delta t}{\tau_c B}} \tag{223}$$

and

$$T_2' = \frac{\frac{q_{net_o} \Delta t}{B} + T_2 + T_3' \frac{k_c \Delta t}{\tau_c B}}{1 + \frac{k_c \Delta t}{\tau_c B}} \quad . \tag{224}$$

2. Cylinder

a. General Thick (Figure 22, T_m Taking on Value T_1') with $\delta_a > \tau_a$

T_3' through T_n', including all interfaces, are calculated from the general heat conduction equations for cylinders.

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is

$$q_{net_o} A_1 - q_{\underset{1 \to 2}{cond}} A_2 = q_{\underset{1}{stored}} A_3 \tag{225}$$

or

$$q_{net_o} A_1 - \frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) A_2$$

$$= A_3 \rho_a c_a \left(\frac{\tau_a}{2} - a\right) \frac{\left(T_1' - T_1\right)}{\Delta t} \tag{226}$$

where

$$A_1 = \theta L \left(R - a\right)$$

$$A_2 = \theta L \left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)$$

$$A_3 = \theta L \left(R - \sum \tau_2 + \frac{\frac{3\tau_a}{2} - a}{2}\right).$$

The energy balance for T_2' is the same as Equation (105) if T_m is replaced by T_1'.

Solving Equation (226) and modified Equation (105) simultaneously and letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2}$$

and $Z = R - \sum \tau_2$ gives

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} + \beta_a\left(\frac{B_1}{B_2}\right) + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right)\left(\frac{B_3}{B_1}\right)\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_O}\,\Delta t}{\rho_a\, c_a\, \tau_a}\left(\frac{B_5}{B_4}\right)\right] + \left[\beta_a\left(\frac{B_1}{B_4}\right) T_2 + T_3'\,\beta_a\left(\frac{B_3}{B_1}\right)\right]}{\left[\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right)\right]\left[1 + \beta_a\left(\frac{B_3}{B_1}\right)\right] + \beta_a\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left(\frac{B_1}{B_2}\right)} \tag{227}$$

and

$$T_2' = \frac{\left[\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right)\right]\left[T_2 + T_3'\,\beta_a\left(\frac{B_3}{B_1}\right)\right] + \beta_a\left(\frac{B_1}{B_2}\right)\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_O}\,\Delta t}{\rho_a\, c_a\, \tau_a}\left(\frac{B_5}{B_4}\right)\right]}{\left[\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right)\right]\left[1 + \beta_a\left(\frac{B_3}{B_1}\right)\right] + \beta_a\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left(\frac{B_1}{B_2}\right)} \tag{228}$$

where

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_2 = Z$$

$$B_3 = Z - \frac{\tau_a}{2}$$

$$B_4 = Z + \frac{\frac{3\tau_a}{2} - a}{2}$$

$$B_5 = R - a \quad .$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance for T_1' is

$$q_{net_O}\, A_1 - \underset{1 \rightarrow 2}{q_{cond}}\, A_2 = \underset{1}{q_{stored}}\, A_3 \tag{229}$$

or

$$q_{net_O}\, A_1 + \frac{k_a}{\left(\tau_a - a\right)}\left(T_2' - T_1'\right) A_2 = 0 \tag{230}$$

where

$$A_1 = \theta L\,(R - a)$$

$$A_2 = \theta L\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right) \quad .$$

The energy balance for T_2' is the same as Equation (108) if T_m is replaced by T_1'.

Solving Equation (230) and modified Equation (108) simultaneously and letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2}$$

and $Z = R - \sum \tau_2$ results in

$$T_1' = \frac{q_{net_o}\left(\frac{B_5}{B_1}\right)\left(\frac{\tau_a}{k_a}\right)\left[\frac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right) + \beta_a\left(\frac{\tau_a - a}{\tau_a}\right)\left(\frac{B_3}{B_4}\right)\right] + T_2\left(\frac{\frac{3\tau_a}{2} - a}{\tau_a}\right) + T_3'\beta_a\left(\frac{B_3}{B_4}\right)}{\frac{\frac{3\tau_a}{2} - a}{\tau_a} + \beta_a\left(\frac{B_3}{B_4}\right)}$$

and

$$T_2' = \frac{q_{net_o}\left(\frac{B_5}{B_4}\right)\left(\frac{\tau_a}{k_a}\right)\beta_a + T_2\left(\frac{\frac{3\tau_a}{2} - a}{\tau_a}\right) + T_3'\beta_a\left(\frac{B_3}{B_4}\right)}{\frac{\frac{3\tau_a}{2} - a}{\tau_a} + \beta_a\left(\frac{B_3}{B_4}\right)} \qquad (232)$$

where

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_3 = Z - \frac{\tau_a}{2}$$

$$B_4 = Z + \frac{\frac{\tau_a}{2} - a}{2}$$

$$B_5 = R - a \quad .$$

b. <u>Special Thick (Exposed Surface $\leq \tau$ from Backside) (Figure 23, T_m Taking on Value T_1') with $\delta_a \leq \tau_a$</u>

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is the same as Equation (226).

The energy balance for T_2' is the same as Equation (111) if T_m is replaced by T_1'.

Solving Equation (226) and modified Equation (111) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and $Z = R - \sum \tau_2$ gives

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} + 2\beta_a\left(\frac{B_1}{B_4}\right)\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o}\,\Delta t}{\rho_a\, c_a\, \tau_a}\left(\frac{B_5}{B_4}\right)\right] + \beta_a\left(\frac{B_1}{B_4}\right)\left[T_2 - \frac{2\, q_{net_i}\,\Delta t}{\rho_a\, c_a\, \tau_a}\left(\frac{B_2}{B_6}\right)\right]}{\left[\frac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right)\right] + 2\beta_a\left(\frac{B_1}{B_6}\right)\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} \tag{233}$$

and

$$T_2' = \frac{\left[\frac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right)\right]\left[T_2 - \frac{2\, q_{net_i}\,\Delta t}{\rho_a\, c_a\, \tau_a}\left(\frac{B_2}{B_6}\right)\right] + 2\beta_a\left(\frac{B_1}{B_6}\right)\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o}\,\Delta t}{\rho_a\, c_a\, \tau_a}\left(\frac{B_5}{B_4}\right)\right]}{\left[\frac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right)\right] + 2\beta_a\left(\frac{B_1}{B_6}\right)\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} \tag{234}$$

where

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_2 = Z$$

$$B_4 = Z + \frac{\frac{3\tau_a}{2} - a}{2}$$

$$B_5 = R - a$$

$$B_6 = Z + \frac{\tau_a}{4} \quad .$$

(2) $\frac{\tau_a}{2} < a \leq \tau_a$. The energy balance for T_1' is the same as Equation (230).

The energy balance for T_2' is the same as Equation (114) if T_m is replaced by T_1'.

Solving Equation (230) and modified Equation (114) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and $Z = R - \sum \tau_2$ results in

$$T_1' = T_2 + \frac{\tau_a}{(\tau_a - a)} \left[q_{net_o} \left(\frac{\tau_a}{k_a}\right) \left(\frac{B_5}{B_1}\right) \left\{ \beta_a + \left(\frac{\tau_a - a}{\tau_a}\right)^2 \right\} - \frac{q_{net_i} \, \Delta t}{\rho_a \, c_a \, \tau_a} \left(\frac{B_2}{B_1}\right) \right] \qquad (235)$$

and

$$T_2' = T_2 + \frac{\tau_a}{(\tau_a - a)} \left[q_{net_o} \, \beta_a \left(\frac{\tau_a}{k_a}\right) \left(\frac{B_5}{B_1}\right) - \frac{q_{net_i}}{\rho_a \, c_a \, \tau_a} \left(\frac{B_2}{B_1}\right) \right] \qquad (236)$$

where

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_2 = Z$$

$$B_5 = R - a \; .$$

c. Special Thick-Thin (Figure 24, T_m Taking on Value T_1') with $\delta_a \leq \tau_a$

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is the same as Equation (226).

The energy balance for T_2' is the same as Equation (117) if T_m is replaced by T_1'.

Solving Equation (226) and modified Equation (117) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and $Z = R - \sum \tau_2$ gives

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} + \frac{k_a \, \Delta t}{\tau_a} \left(\frac{B_1}{B}\right)\right] \left[T_1 \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_0} \, \Delta t}{\rho_a \, c_a \, \tau_a} \left(\frac{B_5}{B_4}\right)\right] + \beta_a \left(\frac{B_1}{B_4}\right) \left[T_2 - q_{net_1} \, \Delta t \left(\frac{B_7}{B}\right)\right]}{\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a \left(\frac{B_1}{B_4}\right) + \frac{k_a \, \Delta t}{\tau_a} \left(\frac{B_1}{B}\right) \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} \tag{237}$$

and

$$T_2' \quad \frac{\left[\frac{k_a \, \Delta t}{\tau_a} \left(\frac{B_1}{B}\right)\right] \left[T_1 \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_0} \, \Delta t}{\rho_a \, c_a \, \tau_a} \left(\frac{B_5}{B_4}\right)\right] + \left[\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a \left(\frac{B_1}{B_4}\right)\right] \left[T_2 - q_{net_1} \, \Delta t \left(\frac{B_7}{B}\right)\right]}{\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a \left(\frac{B_1}{B_4}\right) + \frac{k_a \, \Delta t}{\tau_a} \left(\frac{B_1}{B}\right) \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)} \tag{238}$$

where

$$B = \left(Z + \frac{\tau_a}{4}\right) \rho_a \, c_a \, \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{2}\right) \rho_b \, c_b \, \tau_b$$

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_4 = Z + \frac{\frac{3\tau_a}{2} - a}{2}$$

$$B_5 = R - a$$

$$B_7 = Z - \tau_b \; .$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$. The energy balance for T_1' is the same as Equation (230).

The energy balance for T_2' is the same as Equation (120) if T_m is replaced by T_1'.

Solving Equation (230) and modified Equation (120) simultaneously and letting $Z = R - \sum \tau_2$ results in

$$T_1' = T_2 + q_{net_o}\left[\Delta t\left(\frac{B_5}{B}\right) + \left(\frac{\tau_a - a}{k_a}\right)\left(\frac{B_5}{B_1}\right)\right] - q_{net_i}\,\Delta t\left(\frac{B_7}{B}\right) \qquad (239)$$

and

$$T_2' = T_2 + \frac{\Delta t\left[q_{net_o}(B_5) - q_{net_i}(B_7)\right]}{B} \qquad (240)$$

where

$$B = \left(Z + \frac{\tau_a - a}{2}\right)\rho_a\, c_a\left(\tau_a - a\right) + \left(Z - \frac{\tau_b}{2}\right)\rho_b\, c_b\, \tau_b$$

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_5 = R - a$$

$$B_7 = Z - \tau_b$$

d. Special Thick-Thick (Figure 25, T_m Taking on Value T_1') With $\delta_a \le \tau_a$

T_3' through T_n' are calculated from the general heat conduction equations for cylinders.

(1) $0 \le a \le \frac{\tau_a}{2}$. The energy balance for T_1' is the same as Equation (226).

The energy balance for T_2' is the same as Equation (123) if T_m is replaced by T_1'.

Solving Equation (226) and modified Equation (123) simultaneously and letting $Z = R - \sum \tau_2$ results in

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} - \frac{\frac{k_a \Delta t}{\tau_a}(B_1) + \frac{k_b \Delta t}{\tau_b}(B_8)\left(\frac{\tau_a - a}{\tau_a}\right)}{B}\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{net_o}\,\Delta t\,(B_5)}{\rho_a\, c_a\, \tau_a\,(B_4)}\right] + \delta_a\left(\frac{B_1}{B_4}\right)\left[T_2 + T_3'\,\frac{k_b\,\Delta t\,(B_8)}{\tau_b\, B}\right]}{\delta_a\left(\frac{B_1}{B_4}\right)\left[1 - \frac{k_b\,\Delta t\,(B_8)}{\tau_b\,(B)}\right] + \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left[\frac{\tau_a - a}{\tau_a} - \frac{\frac{k_a}{\tau_a}\Delta t\,(B_1) + \frac{k_b}{\tau_b}\Delta t\,(B_8)\left(\frac{\tau_a - a}{\tau_a}\right)}{B}\right]}$$

(241)

and

$$T_1' = \frac{\left[\frac{k_a \Delta t\,(B_1)}{\tau_a\, B}\right]\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_{neto}\,\Delta t}{\rho_a\, c_a\, \tau_a}\left(\frac{B_5}{B_4}\right)\right] + \left[\frac{(\tau_a - a)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^2} + \beta_a\left(\frac{B_1}{B_4}\right)\right]\left[T_2 + T_3'\,\frac{k_b\,\Delta t}{\tau_b}\left(\frac{B_8}{B}\right)\right]}{\beta_a\left(\frac{B_1}{B_4}\right)\left[1 + \frac{k_b\,\Delta t\,(B_8)}{\tau_b\,(B)}\right] + \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right)\left[\frac{\tau_a - a}{\tau_a} + \frac{\frac{k_a \Delta t\,(B_1)}{\tau_a} + \frac{k_b\,\Delta t\,(B_8)}{\tau_b}\left(\frac{\tau_a - a}{\tau_a}\right)}{B}\right]} \tag{242}$$

where

$$B = \left(Z + \frac{\tau_a}{4}\right)\rho_a\, c_a\,\frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{4}\right)\rho_b\, c_b\,\frac{\tau_b}{2}$$

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_4 = Z + \frac{\frac{3\tau_a}{2} - a}{2}$$

$$B_5 = R - a$$

$$B_8 = Z - \frac{\tau_b}{2}\ .$$

(2) $\frac{\tau_a}{2} < a \leq \tau_a$. The energy balance for T_1' is the same as Equation (230).

The energy balance for T_2' is the same as Equation (126) if T_m is replaced by T_1'.

Solving Equation (230) and modified Equation (126) simultaneously and letting $Z = R - \sum \tau_2$ gives

$$T_1' = \frac{q_{neto}\left(\frac{\tau_a}{k_a}\right)\left(\frac{B_5}{B_1}\right)\left[\frac{\tau_a - a}{\tau_a} + \frac{\frac{k_a\,\Delta t}{\tau_a}(B_1) + \frac{k_b}{\tau_b}\,\Delta t\,(B_8)\left(\frac{\tau_a - a}{\tau_a}\right)}{(B)}\right] + T_2 + T_3'\,\frac{k_b\,\Delta t\,(B_8)}{\tau_b\,(B)}}{1 + \frac{k_b\,\Delta t\,(B_8)}{\tau_b\,(B)}} \tag{243}$$

and

$$T_2' = \frac{q_{neto}\,\Delta t\left(\frac{B_5}{B}\right) + T_2 + T_3'\,\frac{k_b\,\Delta t\,(B_8)}{\tau_b\,(B)}}{1 + \frac{k_b\,\Delta t\,(B_8)}{\tau_b\,(B)}} \tag{244}$$

where

$$B = \left(Z + \frac{\tau_a - a}{2}\right) \rho_a c_a \left(\tau_a - a\right) + \left(Z - \frac{\tau_b}{4}\right) \rho_b c_b \frac{\tau_b}{2}$$

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_5 = R - a$$

$$B_8 = Z - \frac{\tau_b}{2} \; .$$

e. <u>Special Thick-Thin-Thick (Figure 26, T_m Taking on Value T_1') with $\delta_a \leq \tau_a$</u>

T_3' through T_n' are first calculated using the forward finite-difference equations for cylinders.

(1) <u>$0 \leq a \leq \frac{\tau_a}{2}$</u>. The energy balance for T_1' is the same as Equation (226).

The energy balance for T_2' is the same as Equation (129) if T_m is replaced by T_1'.

Solving Equation (226) and modified Equation (129) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^{\,2}}$$

and $Z = R - \sum \tau_2$ results in

$$T_1' = \frac{\left[\frac{\tau_a - a}{\tau_a} + \frac{k_a \Delta t \,(B_1)}{\tau_a \,(B)} + \frac{k_c \, \Delta t}{\tau_c \,(B)} \,(B_9) \left(\frac{\tau_a - a}{\tau_a}\right)\right] \left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_m t_0 \, \Delta t}{\rho_a \, c_a \, \tau_a} \left(\frac{B_5}{B_4}\right)\right] + \beta_a\left(\frac{B_1}{B_4}\right)\left[T_2 + T_3' \frac{k_c \, \Delta t}{\tau_c} \left(\frac{B_9}{B}\right)\right]}{\beta_a \left(\frac{B_1}{B_4}\right)\left[1 + \frac{k_c \, \Delta t}{\tau_c} \left(\frac{B_9}{B}\right)\right] + \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) \left[\frac{\tau_a - a}{\tau_a} + \frac{k_a \Delta t}{\tau_a} \left(\frac{B_1}{B}\right) + \frac{k_c \, \Delta t}{\tau_c} \left(\frac{B_9}{B}\right)\left(\frac{\tau_a - a}{\tau_a}\right)\right]} \tag{245}$$

and

$$T_2' = \frac{\frac{k_a \, \Delta t}{\tau_a} \left(\frac{B_1}{B}\right)\left[T_1\left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) + \frac{q_m t_0 \, \Delta t}{\rho_a \, c_a \, \tau_a} \left(\frac{B_5}{B_4}\right)\right] + \left[\frac{\left(\tau_a - a\right)\left(\frac{\tau_a}{2} - a\right)}{\tau_a^{\,2}} + \beta_a\left(\frac{B_1}{B_4}\right)\right]\left[T_2 + T_3' \frac{k_c \, \Delta t}{\tau_c} \left(\frac{B_9}{B}\right)\right]}{\beta_a \left(\frac{B_1}{B_4}\right)\left[1 + \frac{k_c \, \Delta t}{\tau_c} \left(\frac{B_9}{B}\right)\right] + \left(\frac{\frac{\tau_a}{2} - a}{\tau_a}\right) \left[\frac{\tau_a - a}{\tau_a} + \frac{k_a \, \Delta t}{\tau_a} \left(\frac{B_1}{B_4}\right) + \frac{k_c \, \Delta t}{\tau_c} \left(\frac{B_9}{B}\right) \left(\frac{\tau_a - a}{\tau_a}\right)\right]} \tag{246}$$

where

$$B = \left(Z + \frac{\tau_a}{4}\right) \rho_a c_a \frac{\tau_a}{2} + \left(Z - \frac{\tau_b}{2}\right) \rho_b c_b \tau_b + \left(Z - \tau_b - \frac{\tau_c}{4}\right) \rho_c c_c \frac{\tau_c}{2}$$

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_4 = Z + \frac{\frac{3\tau_a}{2} - a}{2}$$

$$B_5 = R - a$$

$$B_9 = Z - \tau_b - \frac{\tau_c}{2} \quad .$$

(2) $\frac{\tau_a}{2} < a \leq \tau_a$. The energy balance for T_1' is the same as Equation (230).

The energy balance for T_2' is the same as Equation (132) if T_m is replaced by T_1'.

Solving Equation (230) and modified Equation (132) simultaneously and letting $Z = R - \sum \tau_2$ gives

$$T_1' = \frac{q_{net_o}\left(\frac{\tau_a}{k_a}\right)\left(\frac{B_5}{B_1}\right)\left[\frac{\tau_a - a}{\tau_a} + \frac{k_a\,\Delta t}{\tau_a}\left(\frac{B_1}{B}\right) + \frac{k_c\,\Delta t}{\tau_c}\left(\frac{B_9}{B}\right)\left(\frac{\tau_a - a}{\tau_a}\right)\right] + T_2 + T_3'\frac{k_c\,\Delta t}{\tau_c}\left(\frac{B_9}{B}\right)}{1 + \frac{k_c\,\Delta t}{\tau_c}\left(\frac{B_9}{B}\right)} \tag{247}$$

and

$$T_2' \quad \frac{q_{net_o}\,\Delta t\left(\frac{B_5}{B}\right) + T_2 + T_3'\frac{k_c\,\Delta t}{\tau_c}\left(\frac{B_9}{B}\right)}{1 + \frac{k_c\,\Delta t}{\tau_c}\left(\frac{B_9}{B}\right)} \tag{248}$$

where

$$B = \left(Z + \frac{\tau_a - a}{2}\right) \rho_a c_a \left(\tau_a - a\right) + \left(Z - \frac{\tau_b}{2}\right) \rho_b c_b \tau_b$$

$$+ \left(Z - \tau_b - \frac{\tau_c}{4}\right) \rho_c c_c \frac{\tau_c}{2}$$

$$B_1 = Z + \frac{\tau_a - a}{2}$$

$$B_5 = R - a$$

$$B_9 = Z - \tau_b - \frac{\tau_c}{2} \quad .$$

3. **Sphere**

a. General Thick (Figure 27, T_m Taking on Value T_1') with $\delta_a > \tau_a$

T_3' through T_n', including any interfaces, are calculated from the general heat conduction equations for spheres.

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is

$$q_{net_o} A_1 - q_{\underset{1 \rightarrow 2}{cond}} A_2 = q_{\underset{1}{stored}} A_3 \tag{249}$$

or

$$q_{net_o} A_1 - \frac{k_a}{\left(\tau_a - a\right)} \left(T_1' - T_2'\right) A_2$$
$$= A_3 \rho_a c_a \left(\frac{\tau_a}{2} - a\right) \frac{\left(T_1' - T_1\right)}{\Delta t} \tag{250}$$

where

$$A_1 = \phi \left[(R - a)^2\right]$$

$$A_2 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right]$$

$$A_3 = \phi \left[\left(R - \sum \tau_2 + \frac{\frac{3\tau_a}{2} - a}{2}\right)^2 + \frac{\left(\frac{\tau_a}{2} - a\right)^2}{12}\right] .$$

The energy balance for T_2' is the same as Equation (135) if T_m is replaced by T_1'.

Solving Equation (250) and modified Equation (135) simultaneously and letting $Z = R - \sum \tau_2$ results in Equations (227) and (228) where the following parameters take on the new values of

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \tag{251}$$

$$B_2 = Z^2 + \frac{\tau_a^2}{12} \tag{252}$$

$$B_3 = \left(Z - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12} \tag{253}$$

$$B_4 = \left[\left(Z + \frac{\frac{3\tau_a}{2} - a}{2}\right)^2 + \frac{\left(\frac{\tau_a}{2} - a\right)^2}{12}\right] \tag{254}$$

$$B_5 = (R - a)^2 \ . \tag{255}$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a.}$ The energy balance for T_1' is

$$q_{net_o} A_1 - q_{\underset{1 \to 2}{cond}} A_2 = q_{\underset{1}{stored}} A_3 \tag{256}$$

or

$$q_{net_o} A_1 + \frac{k_a}{(\tau_a - a)} \left(T_2' - T_1'\right) A_2 = 0 \tag{257}$$

where

$$A_1 = \phi (R - a)^2$$

$$A_2 = \phi \left[\left(R - \sum \tau_2 + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] .$$

The energy balance for T_2' is the same as Equation (138) if T_m is replaced by T_1'.

Solving Equation (257) and modified Equation (138) simultaneously and letting $Z = R - \sum \tau_2$ gives Equations (231) and (232) where the following parameters take on the new values of

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \tag{258}$$

$$B_3 = \left[\left(Z - \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \tag{259}$$

$$B_4 = \left[\left(Z + \frac{\frac{\tau_a}{2} - a}{2}\right)^2 + \frac{\left(\frac{3\tau_a}{2} - a\right)^2}{12}\right] \tag{260}$$

$$B_5 = (R - a)^2 \quad . \tag{261}$$

b. Special Thick (Exposed Surface $\leq \tau_a$ from Backside) (Figure 28, T_m Taking on Value T_1') with $\delta_a \leq \tau_a$

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is the same as Equation (250).

The energy balance for T_2' is the same as Equation (141) if T_m is replaced by T_1'.

Solving Equation (250) and modified Equation (141) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and $Z = R - \sum \tau_2$ results in Equations (233) and (234) where the following parameters take on the new values of

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \tag{262}$$

$$B_2 = Z^2 + \frac{\tau_a^2}{12} \tag{263}$$

$$B_4 = \left[\left(Z + \frac{\frac{3\tau_a}{2} - a}{2}\right)^2 + \frac{\left(\frac{\tau_a}{2} - a\right)^2}{12}\right] \tag{264}$$

$$B_5 = (R - a)^2 \tag{265}$$

$$B_6 = \left[\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \quad . \tag{266}$$

(2) $\frac{\tau_a}{2} < a \le \tau_a$. The energy balance for T_1' is the same as Equation (257).

The energy balance for T_2' is the same as Equation (144) if T_m is replaced by T_1'.

Solving Equation (257) and modified Equation (144) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and $Z = R - \sum \tau_2$ results in Equations (235) and (236) where the following parameters take on the new values of

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \tag{267}$$

$$B_2 = Z^2 \tag{268}$$

$$B_5 = (R - a)^2 \, . \tag{269}$$

c. Special Thick-Thin (Figure 29, T_m Taking on Value T_1') with $\delta_a \le \tau_a$

(1) $0 \le a \le \frac{\tau_a}{2}$. The energy balance for T_1' is the same as Equation (250).

The energy balance for T_2' is the same as Equation (147) if T_m is replaced by T_1'.

Solving Equation (250) and modified Equation (147) simultaneously and letting $Z = R - \sum \tau_2$ gives Equations (237) and (238) where the following parameters take on the new values of

$$B = \left[\left\{\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right\}\rho_a \, c_a \, \frac{\tau_a}{2} + \left\{\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right\} \rho_b \, c_b \, \tau_b\right] \tag{270}$$

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right] \tag{271}$$

$$B_4 = \left[\left(Z + \frac{\frac{3\tau_a}{2} - a}{2}\right)^2 + \frac{\left(\frac{\tau_a}{2} - a\right)^2}{12}\right] \tag{272}$$

$$B_5 = (R - a)^2 \tag{273}$$

$$B_7 = (Z - \tau_b)^2 \quad . \tag{274}$$

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$. The energy balance for T_1' is the same as Equation (257).

The energy balance for T_2' is the same as Equation (150) if T_m is replaced by T_1'.

Solving Equation (257) and modified Equation (150) simultaneously and letting $Z = R - \sum \tau_2$ gives Equations (239) and (240) where the following parameters take on the new values of

$$B = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right]\rho_a\, c_a \left(\tau_a - a\right)$$

$$+ \left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \rho_b\, c_b\, \tau_b \tag{275}$$

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right] \tag{276}$$

$$B_5 = (R - a)^2 \tag{277}$$

$$B_7 = (Z - \tau_b)^2 \quad . \tag{278}$$

d. <u>Special Thick-Thick (Figure 30, T_m Taking on Value T_1') with $\delta_a \le \tau_a$</u>

T_3' through T_n' are calculated from the general heat conduction equations for spheres.

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$. The energy balance for T_1' is the same as Equation (250).

The energy balance for T_2' is the same as Equation (153) if T_m is replaced by T_1'.

Solving Equation (250) and modified Equation (153) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and $Z = R - \sum \tau_2$ results in Equations (241) and (242) where the following parameters take on the new values of

$$B = \left[\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \rho_a \, c_a \, \frac{\tau_a}{2}$$

$$+ \left[\left(Z - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right] \rho_b \, c_b \, \frac{\tau_b}{2} \tag{279}$$

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \tag{280}$$

$$B_4 = \left[\left(Z + \frac{\frac{3\tau_a}{2} - a}{2}\right)^2 + \frac{\left(\frac{\tau_a}{2} - a\right)^2}{12}\right] \tag{281}$$

$$B_5 = (R - a)^2 \tag{282}$$

$$B_8 = \left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] . \tag{283}$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$. The energy balance for T_1' is the same as Equation (257).

The energy balance for T_2' is the same as Equation (156) if T_m is replaced by T_1'.

Solving Equation (257) and modified Equation (156) simultaneously and letting $Z = R - \sum \tau_2$ gives Equations (243) and (244) where the following parameters take on the new values of

$$B = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \rho_a \, c_a \left(\tau_a - a\right)$$

$$+ \left[\left(Z - \frac{\tau_b}{4}\right)^2 + \frac{\tau_b^2}{48}\right] \rho_b \, c_b \, \frac{\tau_b}{2} \tag{284}$$

$$B_1 = \left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12} \tag{285}$$

$$B_5 = (R - a)^2 \tag{286}$$

$$B_8 = \left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \; . \tag{287}$$

e. Special Thick-Thin-Thick (Figure 31, T_m Taking on Value T_1') with $\delta_a \leq \tau_a$

T_3' through T_n' are first calculated using the forward finite-difference equations for spheres.

(1) $0 \leq a \leq \frac{\tau_a}{2}$. The energy balance for T_1' is the same as Equation (250).

The energy balance for T_2' is the same as Equation (159) if T_m is replaced by T_1'.

Solving Equation (250) and modified Equation (159) simultaneously and letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2}$$

and $Z = R - \sum \tau_2$ results in Equations (245) and (246) where the following parameters take on the new values of

$$B = \left[\left(Z + \frac{\tau_a}{4}\right)^2 + \frac{\tau_a^2}{48}\right] \rho_a \, c_a \frac{\tau_a}{2} + \left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \rho_b \, c_b \, \tau_b$$

$$+ \left[\left(Z - \tau_b - \frac{\tau_c}{4}\right)^2 + \frac{\tau_c^2}{48}\right] \rho_c \, c_c \frac{\tau_c}{2} \tag{288}$$

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right] \tag{289}$$

$$B_4 = \left[\left(Z + \frac{\frac{3\tau_a}{2} - a}{2}\right)^2 + \frac{\left(\frac{\tau_a}{2} - a\right)^2}{12}\right] \tag{290}$$

$$B_5 = (R - a)^2 \tag{291}$$

$$B_9 = \left[\left(Z - \tau_b - \frac{\tau_c}{2}\right)^2 + \frac{\tau_c^2}{12}\right] . \tag{292}$$

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$. The energy balance for T_1' is the same as Equation (257).

The energy balance for T_2' is the same as Equation (162) if T_m is replaced by T_1'.

Solving Equation (257) and modified Equation (162) simultaneously and letting $Z = R - \sum \tau_2$ results in Equations (247) and (248) where the following parameters take on the new values of

$$\begin{aligned} B = &\left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right] \rho_a c_a \left(\tau_a - a\right) \\ &+ \left[\left(Z - \frac{\tau_b}{2}\right)^2 + \frac{\tau_b^2}{12}\right] \rho_b c_b \tau_b + \left[\left(Z - \tau_b - \frac{\tau_c}{4}\right)^2\right. \\ &\left.+ \frac{\tau_c^2}{48}\right] \rho_c c_c \frac{\tau_c}{2} \end{aligned} \tag{293}$$

$$B_1 = \left[\left(Z + \frac{\tau_a - a}{2}\right)^2 + \frac{\left(\tau_a - a\right)^2}{12}\right] \tag{294}$$

$$B_5 = (R - a)^2 \tag{295}$$

$$B_9 = \left[\left(Z - \tau_b - \frac{\tau_c}{2}\right)^2 + \frac{\tau_c^2}{12}\right] . \tag{296}$$

Section V. CRITERIA TO STOP ABLATION OR SURFACE RECESSION

Once a material surface has reached the critical temperature for ablating, melting, or subliming, and the surface starts to recede, the criteria for stopping the ablation (recession) must be established. Under normal conditions the heating rate to the ablating surface is decreasing with time when the ablation stops. With this in mind, it was decided to examine the T_1' equations applicable to post-ablation heat flow to see if a criterion could be established. Since q_{net_o} is the only driving parameter found in the T_1' equations in Section IV, there are critical values of q_{net_o} below which the exposed surface temperature cannot be maintained at the ablating temperature. That is, more heat is being conducted internally from the heated surface than is available at the heated surface. The following list shows how to find the net heating rate at which ablation ceases for all structural arrangements considered in this report. In each equation listed, T_1' is set equal to T_m before solving for the critical value of q_{net_o}.

1. Flat Plate

a. General Thick

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Equation (168) solved for q_{net_o}. (297)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Equation (174) solved for q_{net_o}. (298)

b. Special Thick

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Equation (181) solved for q_{net_o}. (299)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Equation (187) solved for q_{net_o}. (300)

c. Special Thick-Thin

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Equation (193) solved for q_{net_o}. (301)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Equation (199) solved for q_{net_o}. (302)

d. Special Thick-Thick

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Equation (205) solved for q_{net_o}. (303)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Equation (211) solved for q_{net_o}. (304)

e. Special Thick-Thin-Thick

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Equation (217) solved for q_{net_o}. (305)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Equation (223) solved for q_{net_o}. (306)

2. Cylinder

a. General Thick

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Equation (227) solved for q_{net_o}. (307)

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$.

Equation (231) solved for q_{net_o}. (308)

b. Special Thick

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$.

Equation (233) solved for q_{net_o}. (309)

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$.

Equation (235) solved for q_{net_o}. (310)

c. Special Thick-Thin

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$.

Equation (237) solved for q_{net_o}. (311)

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$.

Equation (239) solved for q_{net_o}. (312)

d. Special Thick-Thick

(1) $\underline{0 \le a \le \frac{\tau_a}{2}}$.

Equation (241) solved for q_{net_o}. (313)

(2) $\underline{\frac{\tau_a}{2} < a \le \tau_a}$.

Equation (243) solved for q_{net_o}. (314)

e. <u>Special Thick-Thin-Thick</u>

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Equation (245) solved for q_{net_o}. (315)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Equation (247) solved for q_{net_o}. (316)

3. Sphere

a. <u>General Thick</u>

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Solve for q_{net_o} in Equation (227) with Equations (251) through (255) included. (317)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Solve for q_{net_o} in Equation (231) with Equations (258) through (261) included. (318)

b. <u>Special Thick</u>

(1) $0 \leq a \leq \frac{\tau_a}{2}$.

Solve for q_{net_o} in Equation (233) with Equations (262) through (266) included. (319)

(2) $\frac{\tau_a}{2} < a \leq \tau_a$.

Solve for q_{net_o} in Equation (235) with Equations (267) through (269) included. (320)

c. <u>Special Thick-Thin</u>

(1) $\underline{0 \leq a \leq \frac{\tau_a}{2}}$.

Solve for q_{net_o} in Equation (237) with Equations (270) through (274) included. (321)

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$.

Solve for q_{net_o} in Equation (239) with Equations (275) through (278) included. (322)

d. <u>Special Thick-Thick</u>

(1) $\underline{0 \leq a \leq \frac{\tau_a}{2}}$.

Solve for q_{net_o} in Equation (241) with Equations (279) through (283) included. (323)

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$.

Solve for q_{net_o} in Equation (243) with Equations (284) through (287) included. (324)

e. <u>Special Thick-Thin-Thick</u>

(1) $\underline{0 \leq a \leq \frac{\tau_a}{2}}$.

Solve for q_{net_o} in Equation (245) with Equations (288) through (292) included. (325)

(2) $\underline{\frac{\tau_a}{2} < a \leq \tau_a}$.

Solve for q_{net_o} in Equation (247) with Equations (293) through (296) included. (326)

Section VI. CONCLUSIONS

Excellent agreement is obtained between data generated by the combined forward-backward finite-difference equations and selected exact analytical equations for a flat plate undergoing surface recession, when proper time and distance increments are chosen in relation to material thermal properties and surface recession rates. The combined forward-backward finite-difference ablation-conduction method is most accurate when the amount of material removed in a calculation time increment is equal to or less than one-fourth of the selected incremental distance between temperature nodes.

Using the methods presented in this report, simultaneous conduction and ablation calculations for transient, radial heat flow in spheres and cylinders and one-dimensional heat flow in flat plates requires a negligible increase in computer time over a nonreceding case with all other parameters being identical. This is true primarily because the majority of the equations used are identical with those used in nonrecession cases.

Forward-backward finite-difference equations can be mixed to achieve simplicity and to avoid instability in equations for heat flow near the surface of ablating, subliming, or melting structural materials.

Centripetal ablation and heat flow equations for cylind spheres can be modified by minor sign changes to obtain ce ablation and heat flow equations for cylinders and spheres.

Appendix A

DERIVATION OF CENTRIFUGAL HEAT CONDUCTION AND ABLATION EQUATIONS AND A COMPARISON OF THESE EQUATIONS WITH THOSE FOR CENTRIPETAL HEAT FLOW

1. Cylinder

The energy balance for the radial heat flow away from the centerline (centrifugal) of a cylinder (Figure 22) will now be derived for the General Thick case during ablation. For centrifugal flow, R is the inside radius instead of the outside radius used for centripetal flow in Sections I, III, IV, and V of this report.

a. $0 \le a \le \frac{\tau_a}{2}$

The energy balance for T_2 is

$$q_{\underset{1\to 2}{cond}} \; A_{1-2} + q_{\underset{3\to 2}{cond}} \; A_{3-2} = q_{\underset{2}{stored}} \; A_2 \tag{327}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) A_{1-2} + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) A_{3-2}$$

$$= \rho_a \, c_a \, \tau_a \, \frac{(T_2' - T_2)}{\Delta t} \tag{328}$$

where

$$A_{1-2} = \theta L \left[R + \sum \tau_2 - \frac{\tau_a - a}{2}\right] \tag{329}$$

$$A_{3-2} = \theta L \left[R + \sum \tau_2 + \frac{\tau_a}{2}\right] \tag{330}$$

$$A_2 = \theta L \left[R + \sum \tau_2\right] \, . \tag{331}$$

Letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad , \qquad Z = R + \sum \tau_2 \quad ,$$

and solving for T_2'

$$T_2' = \frac{T_m \beta_a \left[\dfrac{Z - \dfrac{\tau_a - a}{2}}{Z}\right] + T_3' \beta_a \left[\dfrac{Z + \dfrac{\tau_a}{2}}{Z}\right] \left(\dfrac{\tau_a - a}{\tau_a}\right) + T_2 \left(\dfrac{\tau_a - a}{\tau_a}\right)}{\beta_a \left[\dfrac{Z - \dfrac{\tau_a - a}{2}}{Z}\right] + \beta_a \left[\dfrac{Z + \dfrac{\tau_a}{2}}{Z}\right] \left(\dfrac{\tau_a - a}{\tau_a}\right) + \dfrac{\tau_a - a}{\tau_a}} \quad . \tag{332}$$

b. $\underline{\dfrac{\tau_a}{2} < a \leq \tau_a}$

The energy balance for T_2 is

$$q_{\underset{1\to 2}{cond}} \underset{1-2}{A} + q_{\underset{3\to 2}{cond}} \underset{3-2}{A} = q_{\underset{2}{stored}} A_2 \tag{333}$$

or

$$\frac{k_a}{\left(\tau_a - a\right)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) \underset{3-2}{A}$$
$$= \rho_a c_a \left(\frac{3\tau_a}{2} - a\right) \frac{\left(T_2' - T_2\right)}{\Delta t} A_2 \tag{334}$$

where

$$\underset{1-2}{A} = \theta L \left[R + \sum \tau_2 - \frac{\tau_a - a}{2}\right] \tag{335}$$

$$\underset{3-2}{A} = \theta L \left[R + \sum \tau_2 + \frac{\tau_a}{2}\right] \tag{336}$$

$$A_2 = \theta L \left[R + \sum \tau_2 - \frac{\tau_a - a}{4}\right] \quad . \tag{337}$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} \quad , \quad Z = R + \sum \tau_2 \quad ,$$

and solving for T_2'

$$T_2' = \frac{T_m \beta_a \left[\dfrac{Z - \dfrac{\tau_a - a}{2}}{Z - \dfrac{\tau_a - 2a}{4}}\right] + T_3' \beta_a \left(\dfrac{\tau_a - a}{\tau_a}\right) \left[\dfrac{Z + \dfrac{\tau_a}{2}}{Z - \dfrac{\tau_a - 2a}{4}}\right] + T_2 \dfrac{\left(\dfrac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}}{\beta_a \left[\dfrac{Z - \dfrac{\tau_a - a}{2}}{Z - \dfrac{\tau_a - 2a}{4}}\right] + \beta_a \left(\dfrac{\tau_a - a}{\tau_a}\right) \left[\dfrac{Z + \dfrac{\tau_a}{2}}{Z - \dfrac{\tau_a - 2a}{4}}\right] + \dfrac{\left(\dfrac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}} \tag{338}$$

Equations (106) and (332) are similar as are Equations (109) and (338). Equation (332) can be obtained from Equation (106) by substituting a multiplication factor into Equation (106). The same is true for Equation (109) to obtain Equation (338). The resulting substitutions and equations are:

$$Z = R - (XQ)\sum \tau_2 \ . \tag{339}$$

c. $\underline{0 \le a \le \frac{\tau_a}{2}}$

$$T_2' = \frac{T_m(\beta_a)\left[\dfrac{Z + (XQ)\frac{\tau_a - a}{2}}{Z}\right] + T_3'\beta_a\left[\dfrac{Z - (XQ)\frac{\tau_a}{2}}{Z}\right]\left(\dfrac{\tau_a - a}{\tau_a}\right) + T_2\left(\dfrac{\tau_a - a}{\tau_a}\right)}{\beta_a\left[\dfrac{Z + (XQ)\frac{\tau_a - a}{2}}{Z}\right] + \beta_a\left[\dfrac{Z - (XQ)\frac{\tau_a}{2}}{Z}\right]\left(\dfrac{\tau_a - a}{\tau_a}\right) + \dfrac{\tau_a - a}{\tau_a}} \tag{340}$$

d. $\underline{\frac{\tau_a}{2} < a \le \tau_a}$

$$T_2' = \frac{T_m(\beta_a)\left[\dfrac{Z + (XQ)\frac{\tau_a - a}{2}}{Z + (XQ)\frac{\tau_a - 2a}{4}}\right] + T_3'\beta_a\left(\dfrac{\tau_a - a}{\tau_a}\right)\left[\dfrac{Z - (XQ)\frac{\tau_a}{2}}{Z + (XQ)\frac{\tau_a - 2a}{4}}\right] + T_2\dfrac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}}{\beta_a\left[\dfrac{Z + (XQ)\frac{\tau_a - a}{2}}{Z + (XQ)\frac{\tau_a - 2a}{4}}\right] + \beta_a\left(\dfrac{\tau_a - a}{\tau_a}\right)\left[\dfrac{Z - (XQ)\frac{\tau_a}{2}}{Z + (XQ)\frac{\tau_a - 2a}{4}}\right] + \dfrac{\left(\frac{3\tau_a}{2} - a\right)\left(\tau_a - a\right)}{\tau_a^2}} \ . \tag{341}$$

Factor XQ takes on the value of +1 for centripetal flow and -1 for centrifugal flow. The appropriate radius is that radius to the original T_1 nodal point. Only one set of equations is compared here; however, all of the equations for the cylindrical flow have been investigated and can be arranged in this form.

Due to the nondimensional terms used, it is also possible to use a -R value for R and achieve the same results. In other words, the centripetal equations presented in the main body of this report can be used in their present form to determine centrifugal heat flow results simply by inserting the radius to the initial inner surface as a negative value.

2. Sphere

The energy balance for the radial flow away from the center in a sphere (Figure 27) will now be derived for the General Thick case during ablation. For centrifugal flow, R is the inside radius instead of the outside radius used for centripetal flow in Sections II, III, IV, and V of this report.

a. $\underline{0 \le a \le \frac{\tau_a}{2}}$

The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} \underset{1-2}{A} + q_{\underset{3 \to 2}{cond}} \underset{3-2}{A} = q_{\underset{2}{stored}} A_2 \tag{342}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) \underset{3-2}{A}$$

$$= \rho_a \, c_a \, \tau_a \, \frac{\left(T_2' - T_2\right)}{\Delta t} \, A_2 \tag{343}$$

where

$$\underset{1-2}{A} = \phi \left[\left(R + \sum \tau_2 - \frac{\tau_a - a}{2} \right)^2 + \frac{(\tau_a - a)^2}{12} \right] \tag{344}$$

$$\underset{3-2}{A} = \phi \left[\left(R + \sum \tau_2 + \frac{\tau_a}{2} \right)^2 + \frac{\tau_a^2}{12} \right] \tag{345}$$

$$A_2 = \phi \left[\left(R + \sum \tau_2 \right)^2 + \frac{\tau_a^2}{12} \right] \tag{346}$$

Letting

$$\beta_a = \frac{k_a \, \Delta t}{\rho_a \, c_a \, \tau_a^2} \quad , \qquad Z = R + \sum \tau_2 \quad ,$$

and solving for T_2'

$$T_2' = \text{[illegible]} \tag{347}$$

b. $\underline{\frac{\tau_a}{2} < a \le \tau_a}$

The energy balance for T_2 is

$$q_{\underset{1 \to 2}{cond}} \underset{1-2}{A} + q_{\underset{3 \to 2}{cond}} \underset{3-2}{A} = q_{\underset{2}{stored}} A_2 \tag{348}$$

or

$$\frac{k_a}{(\tau_a - a)} \left(T_m - T_2'\right) \underset{1-2}{A} + \frac{k_a}{\tau_a} \left(T_3' - T_2'\right) \underset{3-2}{A}$$

$$= \rho_a c_a \left(\frac{3\tau_a}{2} - a\right) \frac{\left(T_2' - T_2\right)}{\Delta t} A_2 \qquad (349)$$

where

$$\underset{1-2}{A} = \phi \left[\left(R + \sum \tau_2 - \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}\right] \qquad (350)$$

$$\underset{3-2}{A} = \phi \left[\left(R + \sum \tau_2 + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}\right] \qquad (351)$$

$$A_2 = \phi \left[\left(R + \sum \tau_2 - \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}\right] . \qquad (352)$$

Letting

$$\beta_a = \frac{k_a \Delta t}{\rho_a c_a \tau_a^2} , \qquad Z = R + \sum \tau_2 ,$$

and solving for T_2'

$$T_2' = \frac{T_m(\beta_a)\left[\dfrac{\left(Z - \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{\left(Z - \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}}\right] + T_3' \beta_a \left(\frac{\tau_a - a}{\tau_a}\right) \left[\dfrac{\left(Z + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{\left(Z - \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}}\right] + T_2 \dfrac{\left(\frac{3\tau_a}{2} - a\right)(\tau_a - a)}{\tau_a^2}}{\beta_a \left[\dfrac{\left(Z - \frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{\left(Z - \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}}\right] + \beta_a \left(\frac{\tau_a - a}{\tau_a}\right) \left[\dfrac{\left(Z + \frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{\left(Z - \frac{\tau_a - 2a}{4}\right)^2 + \frac{(3\tau_a - 2a)^2}{48}}\right] + \dfrac{\left(\frac{3\tau_a}{2} - a\right)(\tau_a - a)}{\tau_a^2}} \qquad (353)$$

Equations (136) and (347) are similar and Equations (139) and (353) are similar. If a multiplication factor is substituted into Equation (136), one obtains Equation (347). The same is true for Equation (139) to obtain Equation (353). The resulting substitutions and equations are:

$$Z = R - (XQ) \sum \tau_2 . \qquad (354)$$

c. $\underline{0 \le a \le \frac{\tau_a}{2}}$

$$T_2' = \frac{T_m(\beta_a)\left[\dfrac{\left(Z - (XQ)\frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right] + T_3(\beta_a)\left[\dfrac{\left(Z + (XQ)\frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right]\left(\frac{\tau_a - a}{\tau_a}\right) + T_2\left(\frac{\tau_a - a}{\tau_a}\right)}{\beta_a\left[\dfrac{\left(Z - (XQ)\frac{\tau_a - a}{2}\right)^2 + \frac{(\tau_a - a)^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right] + \beta_a\left[\dfrac{\left(Z + (XQ)\frac{\tau_a}{2}\right)^2 + \frac{\tau_a^2}{12}}{Z^2 + \frac{\tau_a^2}{12}}\right]\left(\frac{\tau_a - a}{\tau_a}\right) + \dfrac{\tau_a - a}{\tau_a}} . \qquad (355)$$

d. $\dfrac{\tau_a}{2} < a \leq \tau_a$

$$T_i' = \frac{T_m \beta_a \left[\dfrac{\left(Z + (XQ)\dfrac{\tau_a - a}{2}\right)^2 + \dfrac{(\tau_a - a)^2}{12}}{\left(Z + (XQ)\dfrac{\tau_a - 2a}{4}\right)^2 + \dfrac{(3\tau_a - 2a)^2}{48}}\right] + T_i \beta_a \left[\dfrac{\left(Z - (XQ)\dfrac{\tau_a}{2}\right)^2 + \dfrac{\tau_a^2}{12}}{\left(Z + (XQ)\dfrac{\tau_a - 2a}{4}\right)^2 + \dfrac{(3\tau_a - 2a)^2}{48}}\right]\left(\dfrac{\tau_a - a}{\tau_a}\right) + T_2 \dfrac{\left(\dfrac{3\tau_a}{2} - a\right)(\tau_a - a)}{\tau_a^2}}{\beta_a \left[\dfrac{\left(Z + (XQ)\dfrac{\tau_a - a}{2}\right)^2 + \dfrac{(3\tau_a - 2a)^2}{48}}{\left(Z + (XQ)\dfrac{\tau_a - 2a}{4}\right)^2 + \dfrac{(3\tau_a - 2a)^2}{48}}\right] + \beta_a \left[\dfrac{\left(Z - (XQ)\dfrac{\tau_a}{2}\right)^2 + \dfrac{\tau_a^2}{12}}{\left(Z + (XQ)\dfrac{\tau_a - 2a}{4}\right)^2 + \dfrac{(3\tau_a - 2a)^2}{48}}\right]\left(\dfrac{\tau_a - a}{\tau_a}\right) + \dfrac{\left(\dfrac{3\tau_a}{2} - a\right)(\tau_a - a)}{\tau_a^2}} \quad (356)$$

Factor XQ takes on the value of +1 for centripetal flow and -1 for centrifugal flow. For centripetal flow the radius to the original outer surface is used, while for centrifugal flow the radius to the original inner surface is used. Only one set of equations is compared here; however, all of the equations for the spherical flow can be arranged in this form.

Due to the nondimensional terms used, it is also possible to use a -R value for R and achieve the same results. Thus, the centripetal equations presented in the main body of this report for spheres can be used in their present form to calculate centrifugal heat flow effects by inserting the radius to the initial inner surface as a negative value.

Appendix B

COMPARISON OF THEORETICAL TEMPERATURE RESULTS USING FINITE-DIFFERENCE TECHNIQUES FOR FLAT PLATES DURING ABLATION

The designer or analyst always encounters a basic question when using numerical techniques; "How well do the results from these techniques compare with data obtained from exact solutions?" There is only one exact solution that may be readily used to obtain data for comparison with data generated from the forward-backward finite-difference equations described in this report. This exact solution is for a semi-infinite solid, ablating at a constant rate and surface temperature, with the ablated material removed from the surface and swept downstream. This solution also assumes constant thermal properties.

The exact solution is[3]

$$\frac{T_x - T_\infty}{T_m - T_\infty} = e^{-\frac{\dot{a}x}{\alpha}} \tag{357}$$

where

T_x = Temperature at distance x in from the exposed surface.

T_m = Melting, ablating, or subliming temperature.

T_∞ = Temperature at $x = \infty$ from the exposed surface.

$\dot{a}$ = Ablation rate.

x = Distance in from the exposed surface.

α = Thermal diffusivity of the material.

The General Thick equations were used to obtain temperature data for comparison with results from exact solutions. The input values for the exact and finite-difference methods were the following:

ablation rates, $\dot{a}$, = 0.1, 0.25, 0.4, and 0.5 mm/sec.

T_m = 2000°K.

T_∞ = 300°K.

τ_a = 0.001 m = 1.0 mm.

Δt = 1.0 sec.

---

[3]H. S. Carslaw and J. C. Jaeger, CONDUCTION OF HEAT IN SOLIDS, Second Edition, New York, New York, Oxford University Press, 1959.

c_a 0.4 kcal/Kg - °K

k_a 0.36 kcal/m-hr-°K

β_a 0.25

ρ_a 1000 K_g/m^3

α $\frac{k_a}{\rho_a c_a}$ $\beta_a \left(\frac{\tau_a^2}{\Delta t}\right)$ 9×10^{-4} m^2/hr

The steady-state temperature gradient for the finite-difference method was obtained by raising the surface temperature of a semi-infinite slab to the ablating temperature at time zero and calculating the surface recession and temperature distributions for a sufficient time to obtain a steady, nonchanging temperature profile (with respect to distance from the receding surface) in the slab. Comparisons of these steady-state temperature profiles with those obtained from exact solutions are presented in Figures 32 through 35 for four different recession rates. The temperature data comparisons presented in these figures show that the accuracy of the finite-difference method is best when the amount of material removed during a calculation time increment is small in relation to the selected incremental distance between temperature nodes. For example with the β and ablation rate held constant at 0.25 and 0.5 mm/sec respectively, the accuracy of the finite-difference method is improved by decreasing the Δt and τ as shown in Figure 35. This new selection of parameters reduces the r o of $\dot{a}\,\Delta t/\tau$ thereby improving the accuracy of the numerical approximations.

Based on the temperature gradient comparisons in Figures 32 through 35, a parameter may be established as a guide in selecting proper inputs which will result i acceptable finite-difference accuracies. If the finite-difference temperatu deviations shown in Figures 32 and 33 are acceptable and those shown in Figure 34 and solution No. 1 of Figure 35 are not acceptable, for instance, an upper limit of 0.25 for $\dot{a}\,\Delta t/\tau$ is established. Mathematically this criterion is stated as

$$\frac{\dot{a}\,\Delta t}{\tau} \leq 0.25 \qquad (358)$$

where $\dot{a}$ is the maximum expected recession rate.

Equation (358) can be written in another form by considering that $\beta = \frac{k\Delta t}{\rho c \tau^2}$ and $\alpha = \frac{k}{\rho c}$. This equation is

$$\frac{\dot{a}\tau}{\alpha}\beta \leq 0.25 \tag{359}$$

Equation (359) contains the term $\dot{a}\tau/\alpha$ which is equivalent to $\dot{a}x/\alpha$ in Equation (357). If this term is too large the relative temperature difference between T_m and $T_{x=\tau}$ is quite large. Based on the conditions considered in the steady-state temperature comparisons of Figures 32 through 35, an acceptable upper limit for the finite-difference criterion $\dot{a}\tau/\alpha$ is unity. With $\dot{a}\tau/\alpha \leq 1$ the temperature difference between T_m and T_2 (located one τ from T_m) is a maximum of approximately 63 percent of the difference between T_m and the initial equilibrium temperature of the slab.

Satisfying the conditions of $\beta \leq 0.5$, $\dot{a}\Delta t/\tau \leq 0.25$, and $\dot{a}\tau/\alpha \leq 1$ before performing a finite-difference recession-condition analysis aids in insuring that the computed heat transfer in the slab will be reasonably accurate.

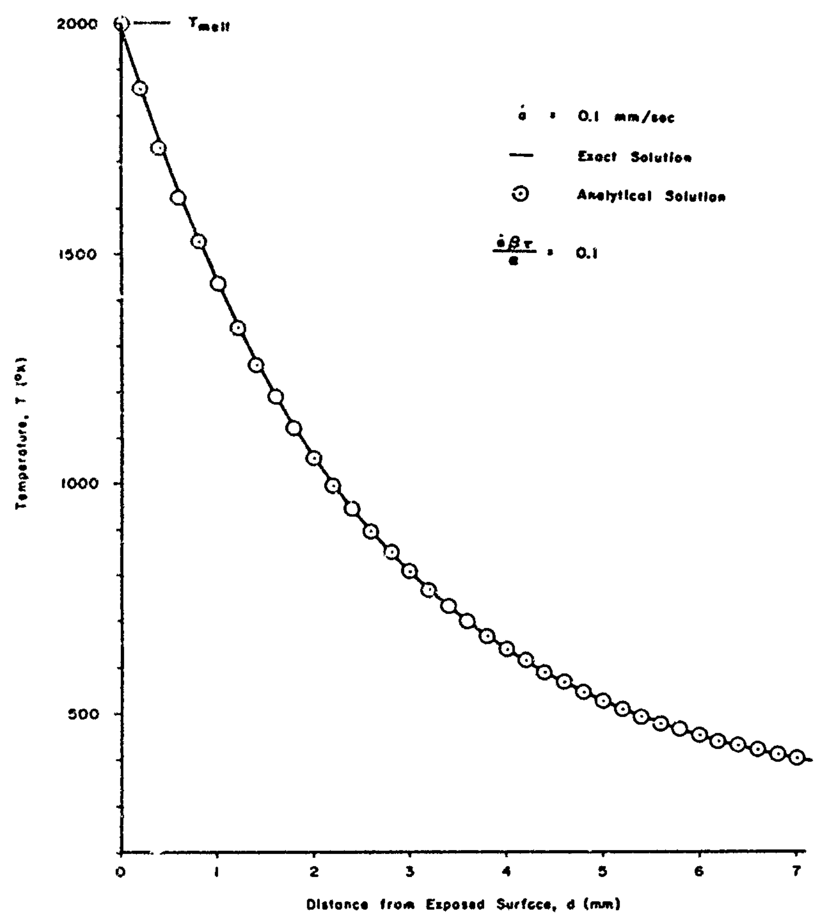


Figure 32.

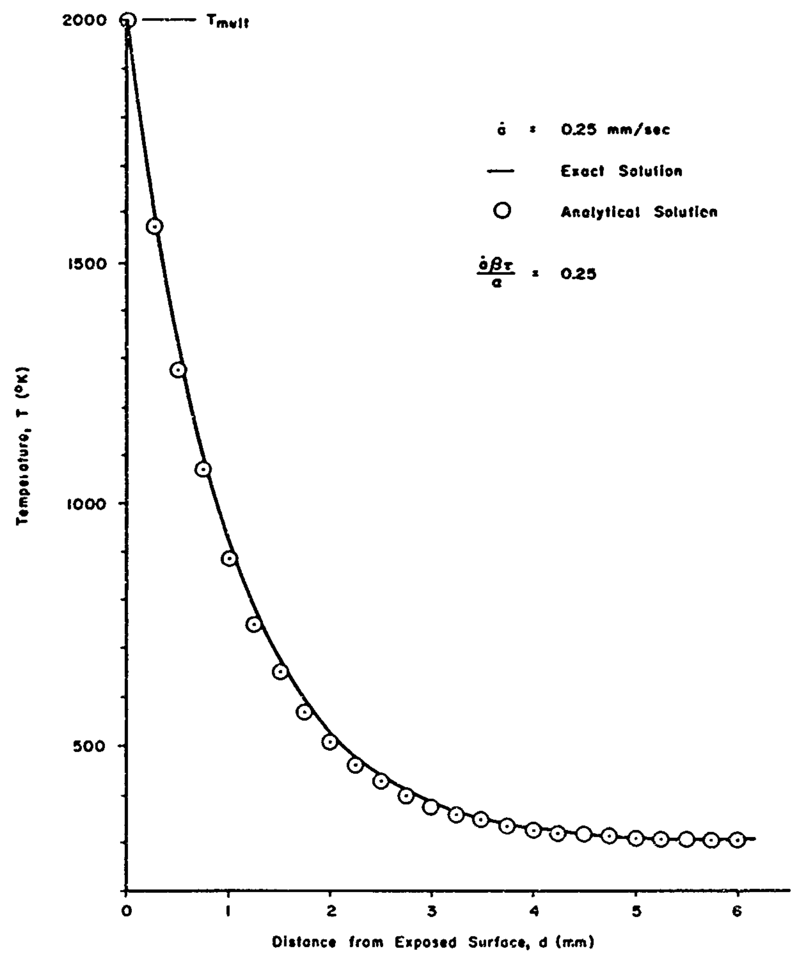


Figure 33.

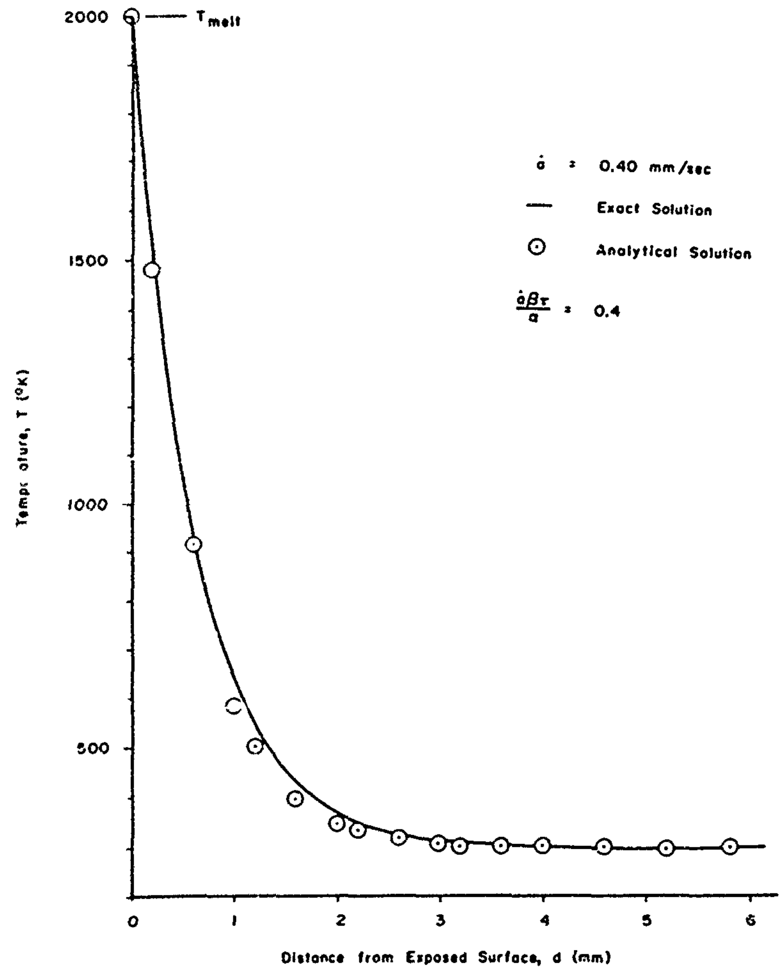


Figure 34.

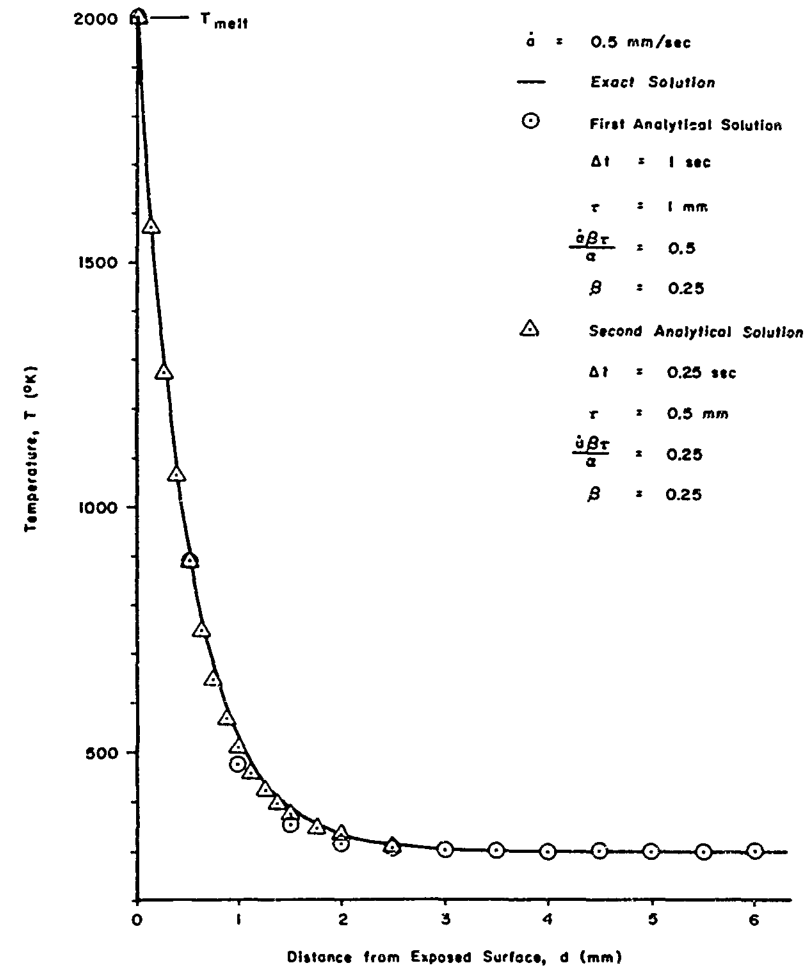


Figure 35.

DISTRIBUTION

	No. of Copies
U. S. Army Missile Command Distribution List A	80
Defense Documentation Center Cameron Station Alexandria, Virginia 22314	20
AMCPM-MA	1
-MB	1
-MB, Mr. Richardson	1
-PE	1
-SE	1
-ZE	1
-ZER, Mr. Webster	1
AMSMI -Y, Mr. Williams	1
-R, Mr. McDaniel	1
Mr. Fagan	1
-RD	1
-RDK	1
-RE	1
-RF	1
-RFE	1
-RG	1
-RH	1
-RHT, Mr. Garner	1
-RK	1
-RL	1
-RN	1
-RNR	1
-RR	1
-RS	1
-RST	25
-RT	1
-RBL	5
-RAP	1

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R&D (Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)	
1. ORIGINATING ACTIVITY (Corporate author) Structures and Mechanics Laboratory, Research and Development Directorate, U. S. Army Missile Command, Redstone Arsenal, Alabama 35809	2a. REPORT SECURITY CLASSIFICATION Unclassified; 2b. GROUP N/A
3. REPORT TITLE SOLUTION OF TRANSIENT HEAT TRANSFER PROBLEMS FOR FLAT PLATES, CYLINDERS, AND SPHERES BY FINITE-DIFFERENCE METHODS WITH APPLICATION TO SURFACE RECESSION	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) None	
5. AUTHOR(S) (Last name, first name, initial) Eppes, R., Jr.	

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1966	112	2

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) RS-TR-66-4
b. PROJECT NO. (DA) 1L013001A91A	
c. AMC Management Structure Code No. 501611844	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AD______
d.	
10. AVAILABILITY/LIMITATION NOTICES This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of this Command, Attn: AMSMI-RS.	
11. SUPPLEMENTARY NOTES None	12. SPONSORING MILITARY ACTIVITY Same as No. 1

13. ABSTRACT

Presented in this report are finite-difference heat-transfer equations for transient, radial heat flow in spheres and cylinders and for transient, one-dimensional heat flow in flat plates. The derived equations apply to structures before, during, and after surface recession for all three basic structure configurations and for several generic material skin combinations.

For each skin configuration the accuracy of the finite-difference procedure, compared with exact analytical methods, depends on optimum selection of the calculation time increment and the incremental distance between temperature nodes in relation to the material thermal properties and on the closeness of the approximate temperature gradients to the true gradients. In addition to these common criteria, the magnitude of the surface recession rate in relation to the calculation time increment and temperature nodal point distance affects the accuracy of the finite-difference temperature results. When compared with exact solutions applicable to semi-infinite flat plates undergoing surface recession, the calculated finite-difference temperature gradients during recession are very accurate when the amount of material removed during a calculation time increment is equal to or less than one fourth of the selected distance increment between temperature nodes.

(Continued on page 115)

DD FORM 1 JAN 64 1473

UNCLASSIFIED
Security Classification

13. Abstract (Concluded)

The cylindrical and spherical equations are presented for centripetal heat flow and surface recession. Two simple methods of converting the centripetal equations to the centrifugal form for applications to structures such as blast tubes, rocket motor combustion chambers, and nozzles are discussed. These two methods involve making a minor number of sign changes in the centripetal heat-flow equations.

Attractive features of the ablation-conduction method described in this report are the negligible increase in required computer time over a nonreceding case when all other parameters are identical. Secondly, the nonshifting temperature grid prevents confusion in interpreting computer results and readily lends itself to automatic plotting techniques.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Transient heat transfer						
Flat plates						
Cylinders						
Spheres						
Finite-difference methods						
Surface recession						
Ablation-conduction method						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.